EE 320 Introductory Mathematical Economics

Semester 2/2012

Problem Set 8 – Suggested Answers

Optimization without Constraint: Multiple-Independent Variable Cases[1]

1.

a) P(10, 8) = 98 and P(12, 10) = 98.

b) FONC: $\pi_x = -2x + 22 = 0$ and $\pi_y = -2y + 18 = 0$. Thus, x* = 11, y*=9, and π* = 100.

2. $x^* = \frac{1}{3}(2p - q - 1)$ and $y^* = \frac{1}{3}(-p + 2q - 1)$, given that $q \geq \frac{1}{2}p + \frac{1}{2}$ and $q \leq 2p - 1$.

3.

a) $x^* = \frac{p}{2\alpha}$ and $y^* = \frac{q}{2\beta}$.

b) $\pi^*(p, q) = \frac{p^2}{4\alpha} + \frac{q^2}{4\beta}$. Hence, $\frac{\partial \pi^*(p,q)}{\partial p} = \frac{p}{2\alpha} = x^*$ and $\frac{\partial \pi^*(p,q)}{\partial q} = \frac{q}{2\beta} = y^*$ Give these results economic interpretations.

4.

a) $\pi_T(p, q) = px + qy - (5 + x) - (3 + 2y) = 26p + 24q - 5p^2 - 6q^2 + 8pq - 69$.
  (p*, q*) = (9, 8) and (x*, y*) = (16, 4).

b) $\pi_A(p) = px - (5 + x) = 34p - 5p^2 + 4pq - 4q - 34$, with q is fixed.

➔ $p = p_A(q) = \frac{1}{5}(2q + 17)$.)

$\pi_B(q) = qy - (3 + 2y) = 28q - 6q^2 + 4pq - 8p - 35$, with p is fixed.

➔ $q = q_B(p) = \frac{1}{3}(p + 7)$.

c) (p*, q*) = (5, 4) and (x*, y*) = (20, 12). $\pi_A = 75$ and $\pi_B = 21$.

5.

a) (i) $R_1'(x_1^*, x_2^*) + s = C_1'(x_1^*, x_2^*)$: Marginal revenue plus subsidy equal marginal cost.

(ii) $R_2'(x_1^*, x_2^*) = C_2'(x_1^*, x_2^*) + t$: Marginal revenue equals marginal cost plus tax.

b) $\pi_{11}'' = R_{11}'' - C_{11}'' < 0$.

$\pi_{11}''\pi_{22}'' - (\pi_{12}'')^2 = (R_{11}'' - C_{11}'')(R_{22}'' - C_{22}'') - (R_{12}'' - C_{12}'')^2 > 0$

[1] All questions in this problem set are from Sydsaeter and Hammond, 2008.

c) Totally differentiating (i) and (ii) gives:

$(R''_{11} - C''_{11})dx_1^* + (R''_{12} - C''_{12})dx_2^* = -ds$, and $(R''_{21} - C''_{21})dx_1^* + (R''_{22} - C''_{22})dx_2^* = dt$.

Solving for dx_1^* and dx_2^* gives:

$dx_1^* = \frac{-(R''_{22}-C''_{22})ds-(R''_{21}-C''_{21})dt}{(R''_{22}-C''_{22})(R''_{11}-C''_{11})-(R''_{12}-C''_{12})^2}$ and $dx_2^* = \frac{(R''_{21}-C''_{21})ds+(R''_{11}-C''_{11})dt}{(R''_{22}-C''_{22})(R''_{11}-C''_{11})-(R''_{12}-C''_{12})^2}$

From the above, we have:

$$\frac{\partial x_1^*}{\partial s} = \frac{-(R''_{22} - C''_{22})}{(R''_{22} - C''_{22})(R''_{11} - C''_{11}) - (R''_{12} - C''_{12})^2} > 0$$

$$\frac{\partial x_1^*}{\partial t} = \frac{-(R''_{21} - C''_{21})}{(R''_{22} - C''_{22})(R''_{11} - C''_{11}) - (R''_{12} - C''_{12})^2} > 0$$

$$\frac{\partial x_2^*}{\partial s} = \frac{(R''_{21} - C''_{21})}{(R''_{22} - C''_{22})(R''_{11} - C''_{11}) - (R''_{12} - C''_{12})^2} < 0$$

$$\frac{\partial x_2^*}{\partial t} = \frac{(R''_{11} - C''_{11})}{(R''_{22} - C''_{22})(R''_{11} - C''_{11}) - (R''_{12} - C''_{12})^2} < 0$$

Where the signs follow from the given assumptions and that $(R''_{22} - C''_{22})(R''_{11} - C''_{11}) - (R''_{12} - C''_{12})^2 > 0$.

d) $\partial x_1^*/\partial t = -\partial x_2^*/\partial s$ because $R''_{12} = R''_{21}$ and $C''_{12} = C''_{21}$.

6.

a) $\pi(Q_1, Q_2) = 120Q_1 + 90Q_2 - 0.1(Q_1^2 + Q_1Q_2 + Q_2^2)$.

FONC: $\pi_1(Q_1, Q_2) = 120 - 0.2Q_1 - 0.1Q_2 = 0$; $\pi_2(Q_1, Q_2) = 90 - 0.1Q_1 - 0.2Q_2 = 0$

Thus, $(Q_1, Q_2) = (500, 200)$ maximizes profit.

b) $\tilde{\pi}(Q_1, Q_2) = P_1Q_1 + 90Q_2 - 0.1(Q_1^2 + Q_1Q_2 + Q_2^2)$

FONC: $\tilde{\pi}_1(Q_1, Q_2) = P_1 - 0.2(400) - 0.1Q_2 = 0$;

$\tilde{\pi}_2(Q_1, Q_2) = 90 - 0.1(400) - 0.2Q_2 = 0$.

It follows that $P_1 = 105$.

7. Define *f(x, y)* for all *(x, y)* by

a) The first-order partial derivatives of *f*:

$f_1'(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}$; $f_2'(x,y) = e^{x+y} - e^{x-y} - \frac{1}{2}$.

The second-order partial derivatives of *f*:

$f_{11}''(x,y) = e^{x+y} + e^{x-y}$; $f_{12}''(x,y) = e^{x+y} - e^{x-y}$; $f_{22}''(x,y) = e^{x+y} + e^{x-y}$

It follows that $f_{11}''(x,y) \geq 0$; $f_{22}''(x,y) \geq 0$; and $f_{11}''f_{22}'' - (f_{12}'')^2 = 4e^{2x} \geq 0$. So, *f* is convex.

b) The stationary point is where $e^{x+y} = 1$ and $e^{x-y} = \frac{1}{2}$, so $x + y = 0$ and $x - y = -ln2$. Thus, the stationary point is $(x,y) = \left(-\frac{1}{2}ln2, \frac{1}{2}ln2\right)$, where $f(x,y) = \frac{1}{2}(1 + ln2)$. Since f is convex, this point is a minimum.

8. a) $f_1'(x,y) = 2x - y - 3x^2$; $f_2'(x,y) = -2y - x$.

$f_{11}''(x,y) = 2 - x$; $f_{12}''(x,y) = -1$; $f_{22}''(x,y) = -2$.

Two stationary points are $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (\frac{5}{6}, -\frac{5}{12})$.

For (0, 0), $f_{11}'' = 2$; $f_{22}'' = -1$; $f_{12}'' = -2$; $f_{11}''f_{22}'' - (f_{12}'')^2 = -5$. Thus,$(x_1, y_1)$ is a saddle point.

For $(\frac{5}{6}, -\frac{5}{12})$, $f_{11}'' = -3$; $f_{22}'' = -1$; $f_{12}'' = -2$; $f_{11}''f_{22}'' - (f_{12}'')^2 = 5$. Thus, (x_2, y_2) is a local maximum point.

b) *f* is concave when $f_{11}'' \leq 0$; $f_{22}'' \leq 0$; and $f_{11}''f_{22}'' - (f_{12}'')^2 \geq 0$. This occurs where $x \geq 5/12$.

9. Consider a firm that produces two different goods A and B. If the total cost function is *C(x, y)* and the prices obtained per unit of A and B are *p* and *q* respectively, then the profit is

$$\pi(x,y) = px + qy - C(x,y) \qquad \text{(i)}$$

a) $p = C_x(x^*, y^*)$ and $q = C_y(x^*, y^*)$.

b) The first-order conditions are:

$$\hat{\pi}_p = pf_p + f + qg_p - C_x f_p - C_y f_p = 0;$$

$$\hat{\pi}_q = pf_q + qg_q + g - C_x f_q - C_y f_q = 0$$

c) $\pi = p(a - bp + cq) + q(\alpha + \beta p - \gamma q) - P(a - bp + cq) - Q(\alpha + \beta p - \gamma q) - R$.

The first-order conditions for maximum-profit:

$$\hat{\pi}_p = a - 2bp + cq + \beta q + bP - \beta Q = 0;$$

$$\hat{\pi}_q = cp + \alpha + \beta p - 2\gamma q - cP - \gamma Q = 0.$$

d) $\frac{\partial^2\hat{\pi}}{\partial p^2} = -2b$; $\frac{\partial^2\hat{\pi}}{\partial q^2} = -2\gamma$, and $\left(\frac{\partial^2\hat{\pi}}{\partial p^2}\right)\left(\frac{\partial^2\hat{\pi}}{\partial q^2}\right) - \left(\frac{\partial^2\hat{\pi}}{\partial p\partial q}\right)^2 = 4\gamma b - (\beta + c)^2$. Thus, we need $4\gamma b \geq (\beta + c)^2$ for SOSC to be satisfied.

e) The first-order conditions:

$$\pi_x = F + xF_x + yG_x - C_x = 0$$

$$\pi_y = xF_y + G + yG_y - C_y = 0$$

f) $\pi(x, y) = x(a - bx - cy) + y(\alpha - \beta x - \gamma y) - Px + Qy + R$.

FONC:

$$\pi_x = a - 2bx - cy - \beta y - P = 0$$

$$\pi_y = -cx + \alpha - \beta x - 2\gamma y - Q = 0$$

g) SOSC: $\frac{\partial^2\pi}{\partial x^2} = -2b$; $\frac{\partial^2\pi}{\partial y^2} = -2\gamma$, and $\left(\frac{\partial^2\pi}{\partial x^2}\right)\left(\frac{\partial^2\pi}{\partial y^2}\right) - \left(\frac{\partial^2\pi}{\partial x\partial y}\right)^2 = 4\gamma b - (\beta + c)^2$. Thus, it follows that $4\gamma b \geq (\beta + c)^2$.