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Question 0: (Three-variable optimization)

Suppose that there are three firms. Each firm produces one single product that is imperfectly substitutable to the other two products, i.e. differentiated products. Each of the firm *strategically* competes with each other, and faces the following market demand equations given by,

$$Q_1 = 80 - 2P_1 + P_{23}$$

$$Q_2 = 80 - 2P_2 + P_{13}$$

$$Q_3 = 80 - 2P_3 + P_{12}$$

where P_{23} is the average of the prices charged by firms 2 and 3, P_{13} is the average of the prices charged by firms 2, P_{12} is the average of the prices charged by firms 1 and 2 [e.g., $P_{12} = 0.5(P_1 + P_2)$]. Suppose that each of the three firms, as indexed by j, has the cost function given by $C^j(Q_j) = c_j Q_j$.

Consider the following problems.

a) Set up the profit function of each individual firm, and derive the profit-maximizing condition of each of the three firms when each of them *optimally* sets for the level of price that maximizes its own profit. (**Hint:** *Derive the best response function of each firm, i.e. optimal contingent plan or reaction function.*)

b) Suppose that Based on (a), what would be the *equilibrium* level of price that each of the firms choose. Calculate the level of profit that each firm yields. (Hint: *Solve for the Bertrand equilibrium with differentiated product*)

c) How does the optimal price vary with respect to the size of marginal cost?

(A.)

$TR = P_1 \cdot Q_1$

$= P_1 (80 - 2P_1 + 0.5(P_2 + P_3))$

$= 80P_1 - 2P_1^2 + 0.5P_1P_2 + 0.5P_1P_3$

$TC = C^1(Q_1) = C_1 + Q_1$

$= C_1 (80 - 2P_1 + 0.5(P_2 + P_3))$

$= 80C_1 - 2P_1C_1 + 0.5P_2C_1 + 0.5P_3C_1$

$\pi = (80P_1 - 2P_1^2 + 0.5P_1P_2 + 0.5P_1P_3) - 80C_1 - 2P_1C_1 + 0.5P_2C_1 + 0.5P_3C_1$

$\frac{\partial \pi_1}{\partial P_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 2C_1$

$TR = P_2 \cdot Q_2$

$= P_2 (80 - 2P_2 + 0.5(P_1 + P_3))$

$= 80P_2 - 2P_2^2 + 0.5P_1P_2 + 0.5P_2P_3$

$TC = C^2(Q_2) = C_2 Q_2$

$= C_2 (80 - 2P_2 + 0.5(P_1 + P_3))$

$= 80C_2 - 2P_2C_2 + 0.5P_1C_2 + 0.5P_3C_2$

$\pi_2 = (80P_2 - 2P_2^2 + 0.5P_1P_2 + 0.5P_2P_3) - (80C_2 - 2P_2 + 0.5P_1C_2 + 0.5P_3C_2)$

$$\frac{\partial \pi_2}{\partial P_2} = 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 2C_2$$

$$TR = P_3 \cdot Q_3$$
$$= P_3(80 - 2P_3 + P_{12})$$
$$= P_3(80 - 2P_3 + 0.5(P_1 + P_2))$$
$$= 80P_3 - 2P_3^2 + 0.5P_1P_3 + 0.5P_2P_3$$

$$TC = C^3(Q_3) = C_3Q_3$$
$$= C_3(80 - 2P_3 + 0.5(P_1 + P_2))$$
$$= 80C_3 - 2P_3C_3 + 0.5P_1C_3 + 0.5P_2C_3$$

$$\pi_3 = (80P_3 - 2P_3^2 + 0.5P_1P_3 + 0.5P_2P_3) - (80C_3 - 2P_3C_3 + 0.5P_1C_3 + 0.5P_2C_3)$$

$$\frac{\partial \pi_3}{\partial P_3} = 80 - 4P_3 + 0.5P_1 + 0.5P_2 + 2C_3$$

$$\frac{\partial \pi_1}{\partial P_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 2C_1 = 0$$

$$\frac{\partial \pi_2}{\partial P_2} = 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 2C_2 = 0$$

$$\frac{\partial \pi_3}{\partial P_3} = 80 - 4P_3 + 0.5P_1 + 0.5P_2 + 2C_3 = 0$$

$$P_1 = \frac{80 + 0.5P_2 + 0.5P_3 + 2C_1}{4} \rightarrow BR_1(P_2, P_3)$$

$$P_2 = \frac{80 + 0.5P_1 + 0.5P_3 + 2C_2}{4} \rightarrow BR_2(P_1, P_3)$$

$$P_3 = \frac{80 + 0.5P_1 + 0.5P_2 + 2C_3}{4} \rightarrow BR_3(P_1, P_2)$$

$$\begin{bmatrix} 4 & -0.5 & -0.5 \\ -0.5 & 4 & -0.5 \\ -0.5 & -0.5 & 4 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 80 + 2C_1 \\ 80 + 2C_2 \\ 80 + 2C_3 \end{bmatrix}$$

$$X = \begin{bmatrix} 26.44 + 0.5142\, C_1 + 0.0734 C_2 + 0.0734 C_3 \\ 26.44 + 0.0734 C_1 + 0.5142\, C_2 + 0.0734\, C_3 \\ 26.44 + 0.0734\, C_1 + 0.0734\, C_2 + 0.5142 C_3 \end{bmatrix}$$

$$P_1^* = 26.44 + 0.5142\, C_1 + 0.0734 C_2 + 0.0734 C_3$$
$$P_2^* = 26.44 + 0.0734 C_1 + 0.5142\, C_2 + 0.0734\, C_3$$
$$P_3^* = 26.44 + 0.0734\, C_1 + 0.0734\, C_2 + 0.5142 C_3$$

Ⓑ
$$P_1^* = 26.44 + 0.5142\, C_1 + 0.0734 C_2 + 0.0734 C_3$$
$$P_2^* = 26.44 + 0.0734 C_1 + 0.5142\, C_2 + 0.0734\, C_3$$
$$P_3^* = 26.44 + 0.0734\, C_1 + 0.0734\, C_2 + 0.5142 C_3$$

$$\pi_1^* = P_1^* \cdot Q_1^*$$
$$= (26.44 + 0.5142 C_1 + 0.0734 C_2 + 0.0734 C_3)(80 - 2P_1 + 0.5(P_2 + P_3))$$

$$\pi_2^* = P_2^* \cdot Q_2^*$$
$$= (26.44 + 0.0734 C_1 + 0.5142 C_2 + 0.0734 C_3)(80 - 2P_2 + 0.5(P_1 + P_3))$$

$$\pi_3^* = P_3^* \cdot Q_3^*$$
$$= (26.44 + 0.0734 C_1 + 0.0734 C_2 + 0.5142 C_3)(80 - 2P_3 + 0.5(P_1 + P_2))$$

c) How does the optimal price vary with respect to the size of marginal cost?

$$\frac{\partial P_1^*}{\partial C_1} = 0.5142 \qquad \frac{\partial P_2^*}{\partial C_2} = 0.5142 \qquad \frac{\partial P_3^*}{\partial C_3} = 0.5142$$

As the MC increase by 1 unit, the optimal price increases by 0.5142 units.

Now suppose that all the three firms agree to operate under a cartel (collusive) agreement. That is, they consider a joint pricing scheme that maximizes the joint profit function of the three firms combined. Consider the following problems.

d) Construct the joint profit function.

$$\Pi = P_1Q_1 + P_2Q_2 + P_3Q_3 - C_1Q_1 - C_2Q_2 - C_3Q_3$$

$$= (80P_1 - 2P_1^2 + 0.5P_1P_2 + 0.5P_1P_3) + (80P_2 - 2P_2^2 + 0.5P_1P_2 + 0.5P_2P_3)$$
$$+ (80P_3 - 2P_3^2 + 0.5P_1P_3 + 0.5P_2P_3) - (80C_1 - 2P_1C_1 + 0.5P_2C_1 + 0.5P_3C_1)$$
$$- (80C_2 - 2P_2C_2 + 0.5P_1C_2 + 0.5P_3C_2) - (80C_3 - 2P_3C_3 + 0.5P_1C_3 + 0.5P_2C_3)$$

e) Given that $c_j = c,\ \forall j$. Calculate the level of optimal price setting that each of the firm will set under the joint profit function problem. Confirm your result with the second-order derivative test.

$$\frac{\partial \Pi}{\partial P_1} = 80 - 4P_1 + 0.5P_2 + 0.5P_3 + 0.5P_2 + 0.5P_3 + 2C_1 - 0.5C_2 - 0.5C_3$$

$$= 80 - 4P_1 + P_2 + P_3 + 2C_1 - 0.5C_2 - 0.5C_3 \quad — ①$$

$$\frac{\partial \Pi}{\partial P_2} = 0.5P_1 + 80 - 4P_2 + 0.5P_1 + 0.5P_3 + 0.5P_3 - 0.5C_1 + 2C_2 - 0.5C_3$$

$$= 80 - 4P_2 + P_1 + P_3 - 0.5C_1 + 2C_2 - 0.5C_3 \quad — ②$$

$$\frac{\partial \Pi}{\partial P_3} = 0.5P_1 + 0.5P_2 + 80 - 4P_3 + 0.5P_1 + 0.5P_2 - 0.5C_1 - 0.5C_2 + 2C_3$$

$$= 80 - 4P_3 + P_1 + P_2 - 0.5C_1 - 0.5C_2 + 2C_3 \quad — ③$$

① = ② ; $4P_1 - P_2 - P_3 = -P_1 + 4P_2 - P_3$

$$P_1 = P_2$$

Put $P_1 = P_2$ into ① & ③

$$4P_1 - P_1 - P_3 = -P_1 - P_1 + 4P_3$$

$$P_1 = P_3$$

Put $P_1 = P_2 = P_3$ into ①

$$4P_1 - P_1 - P_1 = 80 - C$$

$$P_1^* = P_2^* = P_3^* = \frac{80-c}{2} \#$$

S.O.C

$$H = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{bmatrix}$$

$|H_1| = -4$ < 0

$|H_2| = \begin{vmatrix} -4 & 1 \\ 1 & -4 \end{vmatrix} = (-4)(-4) - (1)(1) = 15$ > 0

$|H_3| = \begin{vmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{vmatrix} = -50$ < 0

∴ H is negative definite $d^2\Pi < 0 \rightarrow$ globally concave

f) Is the Cartel agreement self-sustaining? Why?

No, the cartel agreement is not self-sustaining. Since, Qd of each goods are interdependent on one another. If one of them set price a little bit higher than the agreement, the Qd for other products will decrease.

Therefore, the cheated firm will gain relatively higher profit from the others.

g) Revisit question (e) and assume that the marginal costs are varied with respect to firms. What would be the implication of the collusive equilibrium?

If MC are varied with respect to firms, they should set the price of each product based on MC for all goods. If then the price of product 1 has higher MC, holding all other things constant, the firm should sell at relatively.

Therefore, they should satisfy the condition where $MR(Q_1^*) = MR(Q_2^*) = MR(Q_3^*) = M(Q_T)$

Question 2:

Pakorn's utility function depends on the consumption of two commodities, x and y, and it is given by

$$U(x, y) = 2xy$$

Suppose that his income is $72, and the prices per unit of x and y are $4 and $6, respectively. Assume that Pakorn spends all of his income, and the values of x and y are both non-zero.

a. Use the Lagrange method to determine the values of x^* and y^* that maximize Pakorn's utility given an income constraint. Verify that the second-order sufficient conditions are satisfied.

b. Determine the maximum utility level and the Lagrange multiplier. Interpret the economic interpretation of the Lagrange multiplier.

c. Suppose that the income is now $73. Approximate the new maximum utility level.

(a) $\mathcal{L} = 2xy + \lambda(72 - 4x - 6y)$

$\frac{\partial \mathcal{L}}{\partial x} : 2y - 4\lambda = 0 \quad -(1) \quad ; \quad 2y = 4\lambda \Rightarrow \boxed{y = 2\lambda}$

$\frac{\partial \mathcal{L}}{\partial y} : 2x - 6\lambda = 0 \quad -(2) \quad ; \quad 2x = 6\lambda \Rightarrow \boxed{x = 3\lambda}$

Plug-in (3)

$\frac{\partial \mathcal{L}}{\partial \lambda} : 72 - 4x - 6y = 0 \quad -(3)$

(3) ; $72 - 4(3\lambda) - 6(2\lambda) = 0$

$72 - 12\lambda - 12\lambda = 0$

$72 = 24\lambda$

$\lambda^* = 3$

(1) ; $2y - 4(3) = 0 \qquad y^* = 6$

(2) ; $2x - 6(3) = 0 \qquad x^* = 9$

$$H = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{\lambda y} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -4 & -6 \\ -4 & 0 & 2 \\ -6 & 2 & 0 \end{bmatrix} \begin{matrix} 0 & -4 \\ -4 & 0 \\ -6 & 2 \end{matrix}$$

$\bar{H} = (0 + 48 + 48) - (0 + 0 + 0) = 96 > 0$

$\therefore \mathcal{L}$ is positive definite

The function is globally concave and $(x^*, y^*) = (9, 6)$ is globally max. ✱

(b) Lagrange multiplier : $\frac{\partial \mathcal{L}}{\partial I} = \lambda$

$\lambda^* = 3$

It can imply that when an income (I) increases by 1 unit $\mathcal{L}$ will increase by 3 unit #

(c) $\mathcal{L} = 2xy + \lambda(73 - 4x - 6y)$

$\frac{\partial \mathcal{L}}{\partial x} : 2y - 4\lambda = 0$ -(1) ; $2y = 4\lambda \Rightarrow \boxed{y = 2\lambda}$

$\frac{\partial \mathcal{L}}{\partial y} : 2x - 6\lambda = 0$ -(2) ; $2x = 6\lambda \Rightarrow \boxed{x = 3\lambda}$

} Plug-in (3)

$\frac{\partial \mathcal{L}}{\partial \lambda} : 73 - 4x - 6y = 0$ -(3)

(3) ; $73 - 4(3\lambda) - 6(2\lambda) = 0$

$73 - 12\lambda - 12\lambda = 0$

$73 = 24\lambda$

$\lambda^* = \frac{73}{24} = 3.04$

(1) ; $2y - 4\left(\frac{73}{24}\right) = 0$ $y^* = 73/12$

(2) ; $2x - 6\left(\frac{73}{24}\right) = 0$ $x^* = 73/4$

$\therefore (x^*, y^*) = \left(\frac{73}{4}, \frac{73}{12}\right)$ the utility level. #

Question 4:

A consumer has the utility function $U = \ln C + \ln(24 - N)$, where C is consumption and N is labor supply. Her budget constraint is $pC = \bar{M} + wN$, where p is the price of the consumption good, w the wage rate, and $\bar{M}$ the consumer's non-wage income.

(a) Formulate the problem of utility maximization subject to the budget constraint, and derive the first-order conditions, using the Lagrange multiplier approach and ignoring the nonnegativity constraints.

(b) Find the demand function $C = C^*(p, w, \bar{M})$ and the labor supply function $N = N^*(p, w, \bar{M})$ (i.e., express C and N in terms of p, w, and $\bar{M}$). Show that N^* and C^* are homogeneous of degree zero in $(p, w, \bar{M})$.

(c) Let $U^* = \ln[C^*(p, w, \bar{M})] + \ln[24 - N^*(p, w, \bar{M})]$. Show that $\partial U^*/\partial \bar{M} > 0$ and $\partial U^*/\partial p < 0$. Show that U^* is concave in $\bar{M}$ and convex in p. What is the relationship between $\partial U^*/\partial \bar{M}$ and the Lagrange multiplier?

$$U = \underbrace{\ln C + \ln(24 \cdot N)}_{\text{objective function}} + \lambda \underbrace{(pC = \bar{M} + WN)}_{\text{(constant}}$$

$$L(c, N, \lambda; \bar{M}, W, P) = \ln C + \ln(24 - N) + \lambda(\bar{M} + WN - PC)$$

a) $[N]$ $\dfrac{\partial L}{\partial N} = \dfrac{1}{24 - N} \cdot (-1) + \lambda W = \dfrac{-1}{24 - N} + \lambda W = 0$ ①

$[C]$ $\dfrac{\partial L}{\partial C} = \dfrac{1}{C} - \lambda P = 0$ ②

$[\lambda]$ $\dfrac{\partial L}{\partial \lambda} = \bar{M} + WN - PC = 0$ ③

$$\frac{MU_N}{MU_c} = \frac{MP_N}{MP_c}$$

$\dfrac{①}{②} \Rightarrow \dfrac{-1}{24 - N} \cdot C = \dfrac{-W}{P}$

$$\frac{-C}{24 \cdot N} = \frac{-W}{P}$$

$$C^* = \frac{24W - NW}{P}$$

③ ⇒ $\bar{M} + WN - 24W + NW = 0$

$$2NW = -\bar{M} + 24W$$

$$N^* = \frac{-\bar{M} + 24W}{2W}$$

plug in N^* into ③

$$\bar{M} + \left(\frac{-\bar{M} + 24W}{\cancel{2W}}\right)\cancel{W} - PC = 0$$

$$\bar{M} - \frac{\bar{M}}{2} + 12W - PC = 0$$

$$\therefore C^* = \frac{0.5\bar{M} + 12W}{P}$$

$$U = \ln C + \ln(24 - N)$$

$$U^* = \ln\left(\frac{0.5\bar{M} + 12W}{P}\right) + \ln\left(24 + \frac{\bar{M} - 24W}{2W}\right)$$

ⓑ demand function

$$C^* = \frac{0.5\bar{M} + 12W}{P} = \frac{(0.5t\bar{M}) + (12tW)}{(tP)}$$

$$= \frac{(t^1)(0.5\bar{M}) + (12W)}{P}$$

supply function

$$N^* = \frac{-\bar{M} + 24W}{2W} = \frac{(-t\bar{M} + (24tW)}{(2tW)}$$

$$= (t^1)\left(\frac{-\bar{M} + 24W}{2W}\right)$$

c)

$$U^* = \ln\left(\frac{0.5\bar{M}+12W}{P}\right) + \ln\left(24 + \frac{\bar{M}-24W}{2W}\right)$$

$$\frac{\partial U^*}{\partial \bar{M}} = \frac{1}{\frac{0.5\bar{M}+12W}{P}} \cdot \frac{P(0.5) - \cancel{(0.5\bar{M}+12W)(0)}}{P^2} + \frac{1}{\frac{24W+\bar{M}}{2W}} \left(\frac{\cancel{(2W \cdot 1)} - (24W+\bar{M})(0)}{(2W)^{\cancel{2}}} \right)$$

$$= \frac{\cancel{P^2}}{0.5\bar{M}+12W} \cdot \frac{1}{\cancel{P^2}} + \frac{2W}{24+\bar{M}} \cdot \frac{1}{2W} = \frac{1}{24+\bar{M}} \quad > 0$$

$$\frac{\partial U^*}{\partial P} = \frac{1}{\frac{0.5\bar{M}+12W}{P}} \cdot \frac{P(0) - (0.5\bar{M}+12W)(1)}{P^2} + \frac{1}{\frac{24W+\bar{M}}{2W}} \cdot \frac{2W(0) - 24W+\bar{M}(0)}{(2W)^2}$$

$$= \frac{P}{0.5\bar{M}+12W} \cdot \frac{0.5\bar{M}-12W}{P^2} = \frac{-0.5\bar{M}-12W}{0.5\bar{M}+12W} \cdot \frac{1}{P} = \frac{-1}{P} \quad < 0$$

$$\text{S.O.C} \Rightarrow \frac{\partial^2 U}{\partial \bar{M}^2} = \frac{-1}{(24+M)^2} \quad < 0 \quad \rightarrow \text{concave}$$

$$\frac{\partial^2 U}{\partial P^2} = \frac{1}{P^2} \quad > 0 \quad \rightarrow \text{convex}$$

$$\frac{\partial U^*}{\partial M^*} = \frac{1}{24+\bar{M}}$$

mean that when m^* increase by 1 unit, U^* increase by $\frac{1}{24+m}$ units.

Lagrange multiplier $\rightarrow \lambda^*$ $\qquad$ $c^* = \frac{0.5\bar{M}+12W}{P}$

from ② $\Rightarrow \frac{1}{c} - \lambda P = 0$

$$\therefore \lambda^* = \frac{1}{24+\bar{M}}$$

$$\frac{\partial V}{\partial \bar{M}} = \frac{1}{24+\bar{M}}$$

that is when M increase by 1 unit, U increase by $\frac{1}{24+\bar{M}}$ units.