

## Chapter 4 Integration

### Section 4.1 Antiderivatives

**Definition:** A function  $F$  is called an *antiderivative* of  $f$  on an interval  $I$  if  $F'(x) = f(x)$  for all  $x$  in  $I$ .

For instance, let  $f(x) = x^2$ . We know that if  $F(x) = (1/3)x^3$ , then  $F'(x) = x^2 = f(x)$ . But the function  $G(x) = (1/3)x^3 + 75$  also satisfies  $G'(x) = x^2$ . Therefore, both  $F$  and  $G$  are antiderivatives of  $f$ . Indeed, any function of the form  $H(x) = (1/3)x^3 + C$ , where  $C$  is a constant, is an antiderivative of  $f$ .

Thus, if  $F$  and  $G$  are any two antiderivatives of  $f$ , then  $F' = f(x) = G'(x)$  so  $G(x) - F(x) = C$ , where  $C$  is a constant. We can write this as  $G(x) = F(x) + C$ , so we have the following result.

**Theorem:** If  $F$  is an antiderivative of  $f$  on an interval  $I$ , then the most general antiderivative of  $f$  on  $I$  is

$$F(x) + c$$

where  $c$  is an arbitrary constant.

#### Table of Antidifferentiation Formulas

Function	Particular antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n (n \neq -1)$	$(x^{n+1})/(n+1)$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

**Note :** To obtain the most general antiderivative from the particular ones, we have to add a constant (or constants).

Example 1: Find all functions  $f$  such that  $f'(x) = 8x^3 + 12x + 3$ .

Example 2: Find  $f$  where  $f'(x) = 8x^3 + 12x + 3$ ,  $f(1) = 6$ .

## Section 4.2 The Definite Integral

Definition of a Definite Integral :

If  $f$  is a continuous function defined for  $a \leq x \leq b$ , we divide the interval  $[a,b]$  into  $n$  subintervals of equal width  $\Delta x = (b - a)/n$ . We let  $x_0(=a), x_1, x_2, \dots, x_n(=b)$  be the endpoints of these subintervals and we let  $x_1^*, x_2^*, \dots, x_n^*$  be any sample points in these subintervals, so  $x_i^*$  lies in the  $i$ th subinterval  $[x_{i-1}, x_i]$ . Then the definite integral of  $f$  from  $a$  to  $b$  is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

Because we have assumed that  $f$  is continuous, it can be proved that the limit in the definition above always exists and gives the same value no matter how we choose the sample points  $x_i^*$ . If we take the sample points to be right endpoints, then  $x_i^* = x_i$  and the definition of an integral becomes

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x$$

If we choose the sample points to be left endpoints, then  $x_i^* = x_{i-1}$  and the definition becomes

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i-1})\Delta x$$

Alternatively, we could choose  $x_i^*$  to be the midpoint of the subinterval or any other number between  $x_{i-1}$  and  $x_i$ .

Note 1: The symbol  $\int$  was introduced by Leibniz and is called an integral sign. It is an elongated  $S$  and was chosen because an integral is a limit of sums. In the notation

$\int_a^b f(x)dx$ ,  $\int f(x)dx$  is called the **integrand** and  $a$  and  $b$  are called the **limits of integration**;  $a$

is the **lower limit** and  $b$  is the **upper limit**. The symbol  $dx$  has no official meaning by itself;

$\int_a^b f(x)dx$  is all one symbol. The procedure of calculating an integral is called integration.

Note 2: The definite integral  $\int_a^b f(x)dx$  is a number; it does not depend on  $x$ . In fact, we could use any letter in place of  $x$  without changing the value of the integral.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Note 3: The sum

$$\sum_{i=1}^n f(x_i^*)\Delta x$$

that occurs in the definition above is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826-1866). We know that if  $f$  happens to be positive, the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see figure 4.1(a)). By comparing the definition above with the definition of area, we see that the definite integral

$\int_a^b f(x)dx$  can be interpreted as the area under the curve  $y = f(x)$  from  $a$  to  $b$ . (See figure 4.1(b)).

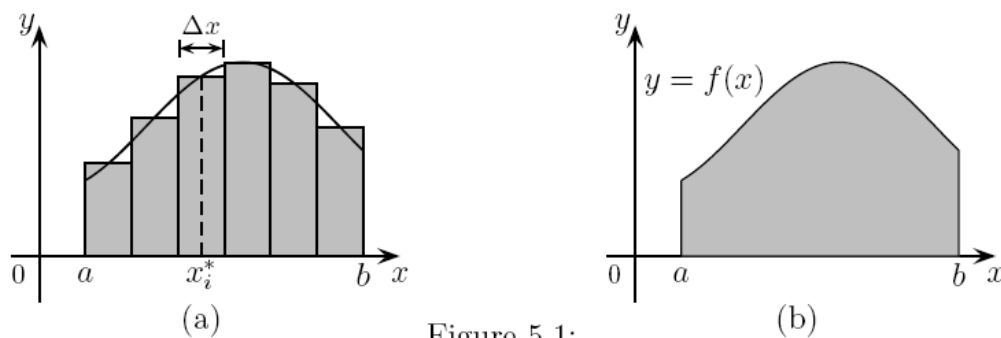
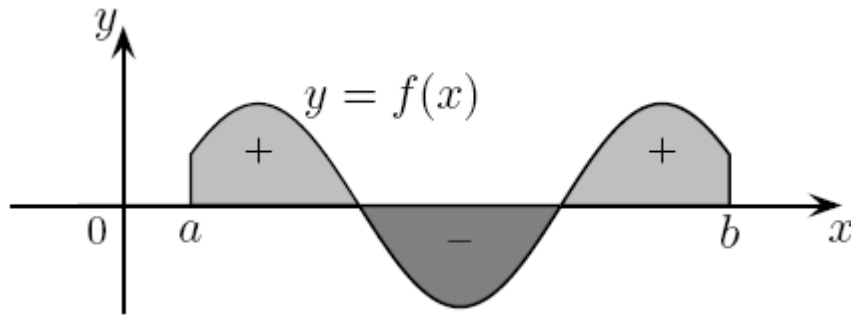


Figure 5.1:

If  $f$  takes on both positive and negative values then the Riemann sum is the sum of the areas of the rectangles that lie above the  $x$ -axis and the *negatives* of the areas of the rectangles that lie below the  $x$ -axis. When we take the limit of such Riemann sums, we get the situation illustrated in figure below.



A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where  $A_1$  is the area of the region above the  $x$ -axis and below the graph of  $f$ , and  $A_2$  is the area of the region below the  $x$ -axis and above the graph of  $f$ .

Note 4: Although we have defined  $\int_a^b f(x)dx$  by dividing  $[a,b]$  into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

## Properties of the Definite Integral

$$(1) \int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$(2) \int_a^a f(x)dx = 0$$

## Properties of the Integral

$$(1) \int_a^b cdx = c(b-a) \text{ where } c \text{ is any constant}$$

$$(2) \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$(3) \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$(4) \int_a^b cf(x)dx = c \int_a^b f(x)dx \text{ where } c \text{ is any constant}$$

$$(5) \int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

## Comparison Properties of the Integral

$$(6) \text{If } f(x) \geq 0 \text{ for } a \leq x \leq b, \text{ then } \int_a^b f(x)dx \geq 0$$

$$(7) \text{If } f(x) \geq g(x) \text{ for } a \leq x \leq b,$$

$$\text{then } \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

$$(8) \text{If } m \leq f(x) \leq M \text{ for } a \leq x \leq b,$$

$$\text{then } m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

## Section 4.3 The Fundamental Theorem of Calculus

### The Fundamental Theorem of Calculus, Part I (FTC I)

If  $f$  is continuous on  $[a,b]$ , then the function  $g$  defined by

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

is continuous on  $[a,b]$  and differentiable on  $(a,b)$ , and  $g'(x) = f(x)$ .

### The Fundamental Theorem of Calculus, Part II (FTC II)

If  $f$  is continuous on  $[a,b]$ , then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where  $F$  is any antiderivative of  $f$ , that is,  $F'(x) = f(x)$

Basically,

Part I says “derivatives” and “integrals” are opposite (or inverse operations).

Part II gives us (finally) a method to evaluate definite integrals without using the limit.

Example 1: Find the derivative of the function.

$$1. g(x) = \int_0^x \cos t dt \quad 2. g(x) = \int_0^x \sin^2 t dt$$

$$3. g(t) = \int_0^t x^3 dx \quad 4. g(x) = \int_0^{x^4} \sec t dt$$

$$5. y = \int_1^{\cos x} (1 + \sin t) dt \quad 6. y = \int_{1/x^2}^0 \sin^3 t dt$$

Example 2: Evaluate the integral.

$$1. \int_1^2 (x^3 - x^2) dx$$

$$2. \int_{-\pi/2}^{\pi/2} \cos x dx$$

$$3. \int_0^1 (2x-1)^2 dx$$

$$4. \int_{-2}^5 6 dx$$

$$5. \int_{-2}^3 (x^{-4}) dx$$

$$6. \int_0^{\pi/6} \csc \theta \cot \theta d\theta$$

$$7. \int_{-\pi}^{\pi} f(x) dx \quad \text{where } f(x) = \begin{cases} x & \text{if } -\pi \leq x \leq 0 \\ \sin x & \text{if } 0 < x \leq \pi \end{cases}$$

$$8. \int_0^{\pi/3} \frac{1 - \sin^2 x}{\cos^4 x} dx$$

## Section 4.4 Indefinite Integrals

Recall :

$$\text{FTC I : } \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

$$\text{FTC II : } \int_a^b f(x)dx = F(b) - F(a) \text{ where } F'(x) = f(x)$$

The notation  $\int f(x)dx$  is traditionally used for an antiderivative of  $f$  and is called an indefinite integral. Thus  $\int f(x)dx = F(x)$  means  $F'(x) = f(x)$

Note: A definite integral  $\int_a^b f(x)dx$  is a number, whereas an indefinite integral  $\int f(x)dx$  is a function (or family of functions).

### Table of Indefinite Integral

$$(1) \int cf(x)dx = c \int f(x)dx$$

$$(2) \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$(3) \int kdx = kx + C$$

$$(4) \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$(5) \int \sin x dx = -\cos x + C$$

$$(6) \int \cos x dx = \sin x + C$$

$$(7) \int \sec^2 x dx = \tan x + C$$

$$(8) \int \csc^2 x dx = -\cot x + C$$

$$(9) \int \sec x \tan x dx = \sec x + C$$

$$(10) \int \csc x \cot x dx = -\csc x + C$$

$$(11) \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$(12) \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$(13) \int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + C$$

Example : Evaluate

$$1. \int \sqrt[3]{x} dx \quad 2. \int \frac{\sin 2x}{\sin x} dx \quad 3. \int_1^2 \frac{y + 5y^7}{y^3} dy \quad 4. \int_0^{\pi/3} \frac{\sin x + \sin x \tan^2 x}{\sec^2 x} dx \quad 5. \int_0^{3\pi/2} |\sin x| dx$$

## Section 4.5 Integration by Substitution

### Section 4.6 Evaluating Definite Integrals by Substitution

Consider the function  $f(x) = 5(x^2 + 3x)^4(2x + 3)$ . To find the antiderivative  $F(x)$  we want  $F'(x) = f(x)$ , it looks as if the Chain Rule was applied to  $F(x)$  to find  $f(x)$ . Try to reverse it to get  $F(x) = (x^2 + 3x)^5 + C$ . Then  $F'(x) = 5(x^2 + 3x)^4(2x + 3) = f(x)$ .

What about  $f(x) = 3x \sin(x^2)$ ,  $F(x) = ?$

An initial guess might be  $F(x) = -\cos(x^2) + C$ ,  $F'(x) = 2x \sin(x^2) \neq f(x) !!!$

What is missing is a factor of  $3/2$ , however since constant multiplies are not affected by integration, we can simply adjust after the fact.

$$F(x) = -(3/2)\cos(x^2) + C$$

$$F'(x) = (-3/2)(2x)(-\sin(x^2)) = 3x \sin(x^2) = f(x)$$

This technique enables us to integrate function via a “reverse chain rule” by involving a new variable which the integral is re-written.

Example:  $\int x^2 \sin(x^3) dx = ?$

- First check that it does look like a reverse chain rule. ie. Look if the derivative of the “inside” appears on the “outside” (apart from a multiplicative constant).
- Then substitute “ $u$ ” as the entire inside function.

Here  $u = x^3$ , we have to have everything as one variable, so we find  $du/dx = 3x^2 \rightarrow du = 3x^2 dx$

Now substitution

$$\int x^2 \sin(x^3) dx = \int \sin(x^3) x^2 dx = \int \sin(u) \frac{1}{3} du = \frac{1}{3} \int \sin(u) du = -\frac{1}{3} \cos u + C = -\frac{1}{3} \cos x^3 + C$$

The Substitution Rule:

If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 1: Find

1.  $\int \frac{\sin(t^{-2} + 1)}{t^3} dt$

2.  $\int y^3 \sqrt{2y^4 - 1} dy$

The Substitution Rule for Definite Integral:

If  $g'(x)$  is continuous on  $[a,b]$  and  $f$  is continuous on the range  $u = g(x)$ , then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 2: Find  $\int_0^{\pi/2} \sin x \cos x dx$

Example 3: Find

1.  $\int_2^4 \frac{dx}{(2x-3)^2}$

2.  $\int_0^4 \frac{dx}{(2x-3)^2}$

3.  $\int_0^7 \sqrt{4+3x} dx$

4.  $\int_{1/6}^{1/2} \csc(\pi t) \cot(\pi t) dt$

5.  $\int_0^{\pi/2} \cos x \sin(\sin x) dx$

6.  $\int \sin^3 x \cos x dx$

7.  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

8.  $\int x^3 (x^2 + 1)^{99} dx$

9.  $\int \frac{1}{\sqrt{1-4x^2}} dx$

10.  $\int \frac{1}{x^2 + a^2} dx$

11.  $\int \frac{x}{x^4 + 9} dx$