Constrained optimization.

2-variable case (2 variables, one constraint)

max (min $f(x,y)$

s.t. $g(x,y) = c$.

$\mathcal{L}_{(x,y,\lambda)} = f(x,y) + \lambda [c - g(x,y)]$

$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0$ } 1-st order condition.

2-nd order Condition

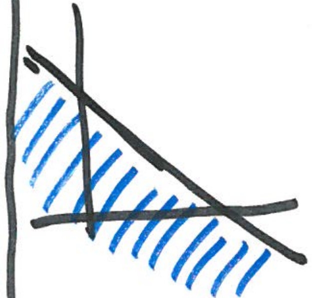
$|\bar{H}| = \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{xy} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$

$|\bar{H}| =$ ⊖ → min, ⊕ → max

⇒ n - choice Variable.

⇒ m - constraint

- Equality Constraint (=) → $x+y=3$
- Inequality Constraint. $\leq$ $\geq$



$x+y \leq 3$

Kuhn - Tucker Method
↳ generalized version of the LaGrange f²

n - Variables
m - equality Constraints

max (min)

$\{x_i\}_{i=1}^{N} : (x_1, x_2, x_3, \ldots, x_N) \longrightarrow$ objective f^{2} $f(x_1, x_2, x_3, \ldots, x_n)$

- Constraint Set: M- Constraint.

$j \Rightarrow 1, 2, 3, \ldots, M$

$\lambda_1 \longrightarrow g^1(x_1, x_2, x_3, \ldots, x_N) = c_1$

$\lambda_2 \longrightarrow g^2(x_1, x_2, x_3, \ldots, x_N) = c_2$

$\vdots$

$\lambda_j \longrightarrow g^j(x_1, x_2, \ldots, x_N) = c_j$

$\vdots$

$\lambda_m \longrightarrow g^m(x_1, x_2, x_3, \ldots, x_N) = c_M$

$n = 2$

$m = 1$

$\mathcal{L} = f(x_1, x_2) + \lambda(c_1 - g^1(x_1, x_2))$

Generalized LaGrange f^{2}

$$\mathcal{L} = f(x_1, x_2, x_3, \ldots, x_N) + \lambda_1[c_1 - g^1(x_1, x_2, \ldots, x_N)] + \lambda_2[c_2 - g^2(x_1, x_2, \ldots, x_N)] + \ldots \lambda_M[c_M - g^m(x_1, x_2, \ldots, x_N)]$$

$$\mathcal{L} = f(x_1, x_2, \dots, x_N) + \sum_{j=1}^{M} \lambda_j \left[C^j - g^j(x_1, x_2, \dots, x_N) \right]$$

$\# N \geqslant \# M$

Theorem: Sol^{n} to the Constraint optimization $\Longleftrightarrow$ $\overset{\text{Stationary}}{\text{Sol}^{n}}$ to the LaGrange f^{n}

$(x_1, x_2, x_3, \dots, x_N, \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_M)$

$$\underbrace{\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \dots = \frac{\partial f}{\partial x_N}}_{N \text{ conditions}} = \frac{\partial f}{\partial \lambda_1} = \frac{\partial \mathcal{L}}{\partial \lambda_2} = \dots = \frac{\partial f}{\partial \lambda_M}$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda_1 \cdot \frac{\partial g^1}{\partial x_1} - \lambda_2 \cdot \frac{\partial g^2}{\partial x_1} - \dots - \lambda_M \cdot \frac{\partial g^M}{\partial x_1}$$

$$= \frac{\partial f}{\partial x_1} - \sum_{j=1}^{M} \lambda_j \cdot \frac{\partial g^j}{\partial x_1} = 0$$

$$[\text{for any } x_i] = \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{M} \cdot \lambda_j \cdot \frac{\partial g^j}{\partial x_i} = 0 \left.\vphantom{\sum_{j=1}^{M}}\right\} = n \text{ conditions}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = c^j - g^j(x_1, x_2, \dots, x_N) = 0 \quad \Big\} \text{ M Conditions.}$$

First-order Condition $\Rightarrow$ N + M Equations.

Theorem: F.O.C.

$$\cancel{\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum \lambda_j \cdot [c^j - \partial}$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{M} \lambda_j \cdot \frac{\partial g^j}{\partial x_i} = 0 \quad \Big\} \text{ N equations}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = c^j - g^j(x_1, x_2, \dots, x_N) = 0 \quad \Big\} \text{ M - equations.}$$

$(x_1^*, x_2^*, \dots, x_N^*, \lambda_1^*, \lambda_2^*, \dots, \lambda_M^*)$ solves for N + M F.O.C. conditions

Bordered heassian

$$\bar{H} = \begin{array}{c} \\ M\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right. \\ N\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right. \end{array} \begin{bmatrix} \mathcal{L}_{\lambda_1\lambda_1} & \mathcal{L}_{\lambda_1\lambda_2} & \cdots & \mathcal{L}_{\lambda_1\lambda_M} & \mathcal{L}_{\lambda_1x_1} & \mathcal{L}_{\lambda_1x_2} & \cdots & \mathcal{L}_{\lambda_1x_N} \\ \mathcal{L}_{\lambda_2\lambda_1} & \mathcal{L}_{\lambda_2\lambda_2} & \cdots & \mathcal{L}_{\lambda_2\lambda_M} & \mathcal{L}_{\lambda_2x_1} & \mathcal{L}_{\lambda_2x_2} & \cdots & \mathcal{L}_{\lambda_2x_N} \\ \vdots & & & & & & & \\ \mathcal{L}_{\lambda_M\lambda_1} & \mathcal{L}_{\lambda_m\lambda_2} & \cdots & \mathcal{L}_{\lambda_m\lambda_m} & \mathcal{L}_{\lambda_mx_1} & \mathcal{L}_{\lambda_mx_2} & \cdots & \mathcal{L}_{\lambda_mx_N} \\ \mathcal{L}_{x_1\lambda_1} & \mathcal{L}_{x_1\lambda_2} & \cdots & \mathcal{L}_{x_1\lambda_m} & \mathcal{L}_{x_1x_1} & \mathcal{L}_{x_1x_2} & \cdots & \mathcal{L}_{x_1x_N} \\ \mathcal{L}_{x_2\lambda_1} & \mathcal{L}_{x_2\lambda_2} & \cdots & \mathcal{L}_{x_2\lambda_m} & \mathcal{L}_{x_2x_1} & \mathcal{L}_{x_2x_2} & \cdots & \mathcal{L}_{x_2x_N} \\ \vdots & & & & & & & \\ \mathcal{L}_{x_n\lambda_1} & \mathcal{L}_{x_n\lambda_2} & \cdots & \mathcal{L}_{x_n\lambda_m} & \mathcal{L}_{x_nx_1} & \mathcal{L}_{x_nx_2} & \cdots & \mathcal{L}_{x_nx_n} \end{bmatrix}$$

Dimension $\Rightarrow (N+M) \times (N+M) \Rightarrow$ Square Matrix

$\bar{H}_k$ = ~~Bo~~ Sub Matrix for the Bordered hessian Matrix.

first k-row
first k-column,

$\bar{H}_1 = \mathcal{L}_{\lambda_1\lambda_1}$ (1×1)

$$\bar{H}_2 = \begin{bmatrix} \mathcal{L}_{\lambda_1\lambda_1} & \mathcal{L}_{\lambda_1\lambda_2} \\ \mathcal{L}_{\lambda_2\lambda_1} & \mathcal{L}_{\lambda_2\lambda_2} \end{bmatrix}_{2\times 2}$$

⋮

$\bar{H}_{10}$ → first 10-row ; first 10-column ⇒ <u>10× 10</u> matrix

$\bar{H}_{N+M}$ → (N+M) The original Bordered Heassian Matrix)

Determinant of the sequence of $|\overline{H}_k|$

small

$\underline{\underline{k}} = 2 \cdot m + 1$ $\quad (m = 1) \Rightarrow H_3$

$\quad m = 4 \Rightarrow H_9$

$|\overline{H}_k|$ sequentially.

$$\begin{array}{ccccccccc} k & \to & k+1 & \to & k+2 & \to & k+3 & \to & \text{last one} \\ \downarrow & & & & & & & & \downarrow \\ 2m+1 & & & & & & & & n+m \end{array}$$

maximum Sol^{n}.

(i) $|\overline{H}_k|$ alternate sign $\forall k$ (ii) first k_{11} $(-1)^{m+1}$

minimum Sol^{n}

(i) $|\overline{H}_k|$ same sign $\forall k$ (ii) first k (smallest sub-matrices) ~~(smallest sub-matrix)~~ all must be $(-1)^{m}$

Suppose that we have 3 unknown of $f(x_1, x_2, x_3)$.

2 Constraint $g^1(x_1, x_2, x_3) = c_1$

$g^2(x_1, x_2, x_3) = c_2$

$M = 2$; $N = 3$

$$\mathcal{L} = f(x_1, x_2, x_3) + \lambda_1 (c_1 - g^1(x_1, \dots, x_3)) + \lambda_2 (c_2 - g^2(x_1, x_2, x_3))$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial \mathcal{L}}{\partial x_3} = \frac{\partial \mathcal{L}}{\partial \lambda_1} = \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0$$ 5 first-order conditions

① x_1 $\quad f_1 - \lambda_1 \cdot g^1_1 - \lambda_2 \cdot g^2_1 = 0$

② x_2 $\quad f_2 - \lambda_1 \cdot g^1_2 - \lambda_2 \cdot g^2_2 = 0$

③ x_3 $\quad f_3 - \lambda_1 \cdot g^1_3 - \lambda_2 \cdot g^2_3 = 0$

④ λ_1 $\quad c_1 - g^1(x_1, x_2, x_3) = 0$

⑤ λ_2 $\quad c_2 - g^2(x_1, x_2, x_3) = 0$

* Sol^n to the problem is $(x_1^*, x_2^*, x_3^*, \lambda_1^*, \lambda_2^*) \Rightarrow$ Solve for the 5 equation at the same time.

$$\bar{H} = \begin{bmatrix} \mathcal{L}_{\lambda_1\lambda_1} & \mathcal{L}_{\lambda_1\lambda_2} & \mathcal{L}_{\lambda_1 x_1} & \mathcal{L}_{\lambda_1 x_2} & \mathcal{L}_{\lambda_1 x_3} \\ \mathcal{L}_{\lambda_2\lambda_1} & \mathcal{L}_{\lambda_2\lambda_2} & \mathcal{L}_{\lambda_2 x_1} & \mathcal{L}_{\lambda_2 x_2} & \mathcal{L}_{\lambda_2 x_3} \\ \mathcal{L}_{x_1\lambda_1} & \mathcal{L}_{x_1\lambda_2} & \mathcal{L}_{x_1 x_1} & \mathcal{L}_{x_1 x_2} & \mathcal{L}_{x_1 x_3} \\ \mathcal{L}_{x_2\lambda_1} & \mathcal{L}_{x_2\lambda_2} & \mathcal{L}_{x_2 x_1} & \mathcal{L}_{x_2 x_2} & \mathcal{L}_{x_2 x_3} \\ \mathcal{L}_{x_3\lambda_1} & \mathcal{L}_{x_3\lambda_2} & \mathcal{L}_{x_3 x_1} & \mathcal{L}_{x_3 x_2} & \mathcal{L}_{x_3 x_3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -g^1_1 & -g^1_2 & g^1_3 \\ 0 & 0 & -g^2_1 & -g^2_2 & -g^2_3 \\ -g^1_1 & -g^2_1 & f_{11}-\lambda_1 \cdot g^1_{11}-\lambda_2 \cdot g^2_{11} & f_{12}-\lambda_1 \cdot g^1_{12}-\lambda_2 \cdot g^2_{12} & f_{13}-\lambda_1 g^1_{13}-\lambda_2 g^2_{13} \\ -s^1_2 & -g^2_2 & f_{12}-\lambda_1 \cdot g^1_{12}-\lambda_2 g^2_{12} & f_{22}-\lambda_1 \cdot g^1_{22}-\lambda_2 \cdot g^2_{22} & f_{23}-\lambda_1 \cdot g^1_{23}-\lambda_2 g^2_{23} \\ -g^1_3 & -g^2_3 & f_{13}-\lambda_1 \cdot g^1_{13}-\lambda_2 \cdot g^2_{13} & f_{23}-\lambda_1 \cdot g^1_{23}-\lambda_2 \cdot g^2_{23} & f_{33}-\lambda_1 \cdot g^1_{33}-\lambda_2 g^2_{33} \end{bmatrix}$$

$\underline{\underline{5 \times 5}}$

maximum Condition

- Smallest size test you needs to check.

$k_{min} = 2 \cdot m + 1$

$\underline{\min k} = 2 \cdot 2 + 1 = 5$

$|\bar{H}_5| \rightarrow$ sign $(-1)^{2+1} < 0$

$|\bar{H}_5| < 0$

min point $|\bar{H}_5|$; $(-1)^2 > 0$

$$\left.\begin{aligned} n &= 5\,(8) \\ m &= 2\,(5) \end{aligned}\right\} \quad \bar{H} \quad 7\times 7$$

$$\begin{aligned} m' \; k &= 2m+1 \\ &= 5 \end{aligned}$$

$$\min k = 2(5)+1 = 11$$

max

$(-1)^6$

$|\bar{H}_{11}| > 0$ $\quad |\bar{H}_{12}| < 0$ $\quad |\bar{H}_{13}| > 0$

$|\bar{H}_5|$ $\quad |\bar{H}_6| > 0$ $\quad |\bar{H}_7| < 0$ $\Big\}$ alternate sign

$(-1)^{m+1}$; $(-1)^3 < 0$

min

$(-1)^5$

$|\bar{H}_{11}| < 0$ $\quad |\bar{H}_{12}| < 0$ $\quad |\bar{H}_{13}| < 0$ $\Big\}$ (Identical same sign)

$|\bar{H}_5|$ $\quad |\bar{H}_6| > 0$ $\quad |\bar{H}_7| > 0$

$(-1)^2 > 0$