Application of Integral

Area

1 Area

Let f be a function defined on the interval $[a, b]$. In this section, we are interested in the following area problems:

- The total area of a region bounded by the graph $y = f(x)$ and the x-axis on an interval $[a, b]$.
- The area of the region bounded between two graphs on an interval $[a, b]$.

Definition 1.1 (Total Area)**.** Suppose the function $y = f(x)$ is continuous on $[a, b]$. The **total area A** bounded by its graph and the x-axis on $[a, b]$ is

$$A = \int_a^b |f(x)| dx.$$

Example 1.1. Find the total area bounded by the graph $y = x^2 - 4$ and the x-axis on $[-1, 3]$.

Example 1.2. Find the total area bounded by the graph $y = \sin(x)$ and the x-axis on $[-\pi, \pi]$.

Definition 1.2 (Area Bounded by Two Graphs)**.** Let f and g be continuous functions on $[a, b]$. The **total area A** bounded by their graphs

$$A = \int_a^b |f(x) - g(x)|dx.$$

Note that

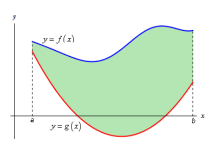
$$|f(x) - g(x)| = \begin{cases} -(f(x) - g(x)), & f(x) - g(x) < 0 \quad \text{or} \quad f(x) < g(x) \\ f(x) - g(x), & f(x) - g(x) \geq 0 \quad \text{or} \quad f(x) \geq g(x). \end{cases}$$

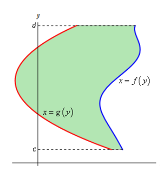
In general, to find the area between two graph, it will be useful to sketch the graphs and see the intersections on the interval $[a, b]$. When $f(x) < g(x)$, the graph of g is *above* the graph of f, and when $f(x) \geq g(x)$, the graph of f is *above* the graph of g.

$$\int_a^b |f(x) - g(x)|dx = \int_a^b [\text{ Upper graph - Lower graph}]\ dx$$

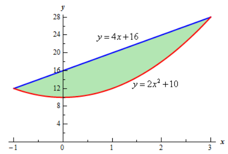
Alternatively, if we can write $x = f(y)$, and $x = g(y)$, then it sometime easier to use

$$\int_c^d |f(y) - f(y)|dy = \int_c^d [\text{ Right graph - Left graph}]\ dy.$$

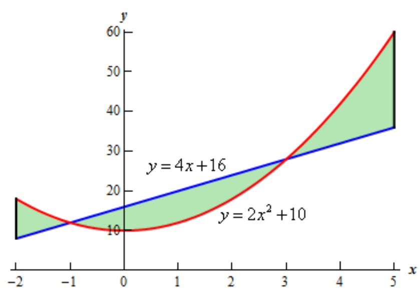




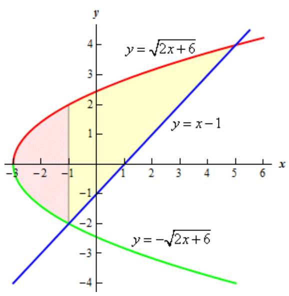
Example 1.3. Determine the area of the region bounded by $2x^2 + 10$ and $y = 4x + 16$.

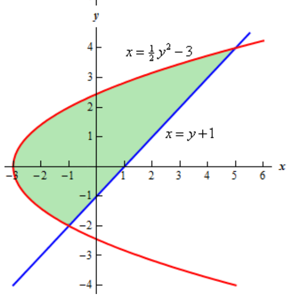


Example 1.4. Determine the area of the region bounded by $2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$.



Example 1.5. Determine the area of the region bounded by $2x = y^2 - 6$ and $y = x - 1$.





Example 1.6. Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$.