CLASS #1

Note Title 2/28/2012

CASH-FLOW MODEL

$$\Delta c_{t+1} = \mu + x_t + \varepsilon^c_{t+1}$$

$$\Delta d_{t+1} = \mu + \lambda_x \cdot x_t + \lambda_c \cdot \varepsilon^c_{t+1} + \varepsilon^d_{t+1}$$

$$x_t = \rho \cdot x_{t-1} + \varepsilon^x_t$$

$$[\varepsilon^c \;\; \varepsilon^x \;\; \varepsilon^d] \sim N\left([0\;0\;0], \begin{bmatrix} \sigma_c^2 & 0 & 0 \\ 0 & \sigma_x^2 & 0 \\ 0 & 0 & \sigma_d^2 \end{bmatrix}\right)$$

SDF: $M_{t+1} = \delta \cdot e^{-\gamma \Delta c_{t+1}}$

$$\hookrightarrow m_{t+1} = \log \delta - \gamma \cdot \mu - \gamma x_t - \varepsilon^c_{t+1}$$

GOAL:

SOLVE FOR • $r^{ex}_{t+1} = r^{m}_{t+1} - r^{f}_{t}$,

• r^{f}_{t},

• $pd_t = \log P/D_t$,

IN A LOG-NORMAL ENVIRONMENT –

STEP 1: RISK-FREE RATE

$$\frac{1}{R^f_t} = E_t[M_{t+1}] = E_t\left[e^{m_{t+1}}\right] = e^{E_t[m_{t+1}] + \frac{1}{2}V_t[m_{t+1}]}$$

TAKE LOGS:

$$-r^f_t = \log\delta - \gamma \cdot \mu - \gamma \cdot x_t + \frac{1}{2} V_t[\Delta c_{t+1}] \cdot \gamma^2$$

$$\downarrow$$

$$r^f_t = \bar{r} + \gamma \cdot x_t \quad \text{WHERE}$$

$$\bar{r} = -\log\delta + \gamma \cdot \mu - \frac{1}{2} \cdot \gamma^2 \sigma_c^2$$

STEP 2: GUESS THAT $pd_t = \overline{pd} + A_1 \cdot x_t$

NA IMPLIES: $1 = E_t\left[e^{m_{t+1}} \cdot R_{t+1}\right]$

WE NEED TO MAKE R_{t+1} APPROXIMATELY LOG-LINEAR

↳ Campbel-Shiller $\log(1+e^{x}) \approx \log(1+e^{\bar{x}}) + \underbrace{\frac{e^{\bar{x}}}{1+e^{\bar{x}}}}_{kd} \cdot (x-\bar{x})$

$$\log R_{t+1} = \log(1+P/D_{t+1}) - \log(P/D_t) + \Delta d_{t+1}$$

$$\approx \ln(1+\bar{P/D}) + kd \cdot \left(\cancel{\overline{pd}} + A_1 \cdot \underbrace{x_{t+1}}_{\rho x_t + \varepsilon^{x}_{t+1}} - \cancel{\overline{pd}}\right) - \overline{pd} - A_1 \cdot x_t + \Delta d_{t+1}$$

$$\approx \ln(1+\bar{P/D}) - \overline{pd} + A_1 \cdot (kd \cdot \rho - 1) \cdot x_t + kd \cdot A_1 \cdot \varepsilon^{x}_{t+1} + \Delta d_{t+1}$$

$$\approx \text{CONST.} + A_1 (kd \cdot \rho - 1) x_t + kd \cdot A_1 \cdot \varepsilon^{x}_{t+1} + \lambda_c \cdot \varepsilon^{c}_{t+1} + \varepsilon^{d}_{t+1}$$
$$+ \lambda_x \cdot x_t$$

USE THE N.A. TO SOLVE FOR A_1:

$$1 = E_t\left[e^{\left(\bar{m} - \gamma \cdot x_t - \gamma \cdot \varepsilon^c_{t+1}\right) + \ln\left(1+\frac{\bar{P}}{D}\right) + A_1 \cdot (\kappa_d \cdot \rho - 1)\, x_t + \kappa_d A_1\, \varepsilon^x_{t+1} + \lambda_c \cdot \varepsilon^c_{t+1} + \varepsilon^d_{t+1} + \lambda_x \cdot x_t} \right]$$

$E_t[\,\cdot\,]$ (1) IS A FUNCTION OF x_t ONLY AS $\varepsilon^c, \varepsilon^x, \varepsilon^d$ ARE i.i.d.

(2) IS CONSTANT AT THE EQUILIBRIUM $(=1)$

$$-\gamma + A_1(\kappa_d \cdot \rho - 1) + \lambda_x = 0 \rightarrow \boxed{A_1 = \frac{\lambda_x - \gamma}{1 - \rho \kappa_d}}$$

STEP 3: SOLVE FOR $K_d = \dfrac{\overline{P/D}}{1+\overline{P/D}}$

USE NA + GUESS + $x_t = E[x_t] = 0$

OUR GUESS IN LEVELS: $\dfrac{P}{D}_t = \overline{P/D} \cdot e^{A_1 \cdot x_t}$

$$\text{NA:}\quad \overline{P/D} \cdot \cancel{e^{A_1 \cdot 0}} = E\left[e^{m_{t+1}} \cdot \left(1+\overline{P/D}\right) \cdot e^{K_d \cdot A_1 \cdot x_{t+1}} \cdot e^{\mu + \lambda_x \cdot x_{t+1} + \lambda_c \cdot \varepsilon^c_{t+1} + \varepsilon^d_{t+1}} \,\middle|\, x_t = 0 \right]$$

$$\frac{\overline{P/D}}{\left(1+\overline{P/D}\right)} = \delta \cdot e^{-\gamma\mu + \mu} \cdot E\left[e^{\left(-\gamma \varepsilon^c_{t+1} + \lambda \varepsilon^c_{t+1}\right) + K_d A_1 \cdot \varepsilon^x_{t+1} + \varepsilon^d_{t+1}} \right]$$

$$K_d = \delta e^{-(\gamma-1)\cdot\mu} \cdot e^{\frac{1}{2}\left[(\gamma-1)^2\sigma_c^2 + (K_d \cdot \underbrace{A_1}_{\frac{\lambda_x-\gamma}{1-\rho K_d}} \cdot \sigma_x)^2 + \sigma_d^2\right]}$$

ONE EQUATION FOR ONE UNKNOWN (K_d) : DONE!

STEP 4 : EQUITY PREMIUM: JUST A COV

$$E\left[r^d_{t+1} - r^f_t\right] = -\text{cov}_t\left(m_{t+1}, r^d_{t+1} - r^f_t\right) \underbrace{- \frac{1}{2} V_t\left(r^d_{t+1}\right)}_{\text{LOG ADJUSTMENT}}$$

LOG UNITS

Risk premium IN LEVELS

⚠ ALL THESE MOMENTS ARE CONDITIONAL AT TIME-t INFO

USING OUR CAMPBELL-SHILLER APPROX + GUESS FOR $\ln(P/D_t)$:

$$r^d_{t+1} = \underbrace{\bar{r}}_{E_t[r^d_{t+1} - r^f_t]} + r^f_t + k_d \cdot A_1 \cdot \varepsilon^x_{t+1} + \lambda_c \cdot \varepsilon^c_{t+1} + \varepsilon^d_{t+1}$$

- $-\mathrm{cov}_t(m_{t+1}, r^{ex}_{t+1}) = -\gamma \cdot \lambda_c \cdot \sigma_c^2$
- $V_t(r^d_t) = V_t(r^{ex}_{t+1}) = (A_1 \cdot k_d \cdot \sigma_x)^2 + \sigma_d^2 + (\lambda_c \cdot \sigma_c)^2$