EE312 Macroeconomic Theory

Chapter 8

A Real Intertemporal Model with Investment (Part I)

State of your knowledge

- In the previous two chapters, we discussed about the two basic settings.

Model 1 - Static **consumption-leisure** equilibrium: ***static production economy***

Model 2 - Intertemporal **consumption-saving** model: ***Intertemporal pure endowment economy***

- How shocks affect the equilibrium in each model

Real Intertemporal Model

- This chapter takes you another step closer to the reality

- We **merge together** the two basic models (model 1 and model 2), and then *newly* introduce an **“investment” decision problem** as an extension
  - Firm needs to think about how much to invest for future

- All these combined features give rise to a ***real intertemporal production economy with capital accumulation***

Investment: why is it important?

- Expenditure on plants, equipment and new housing.
  - Investment goods currently produced for future production of goods and services.

- Increases in ***future productive capacity***.
  - **The consumer's tradeoff** between current and future consumptions (savings).
  - **The firm's tradeoff** between current profits and higher future capital stock (and future profits).

Structure of the real Intertemporal Model: Agents

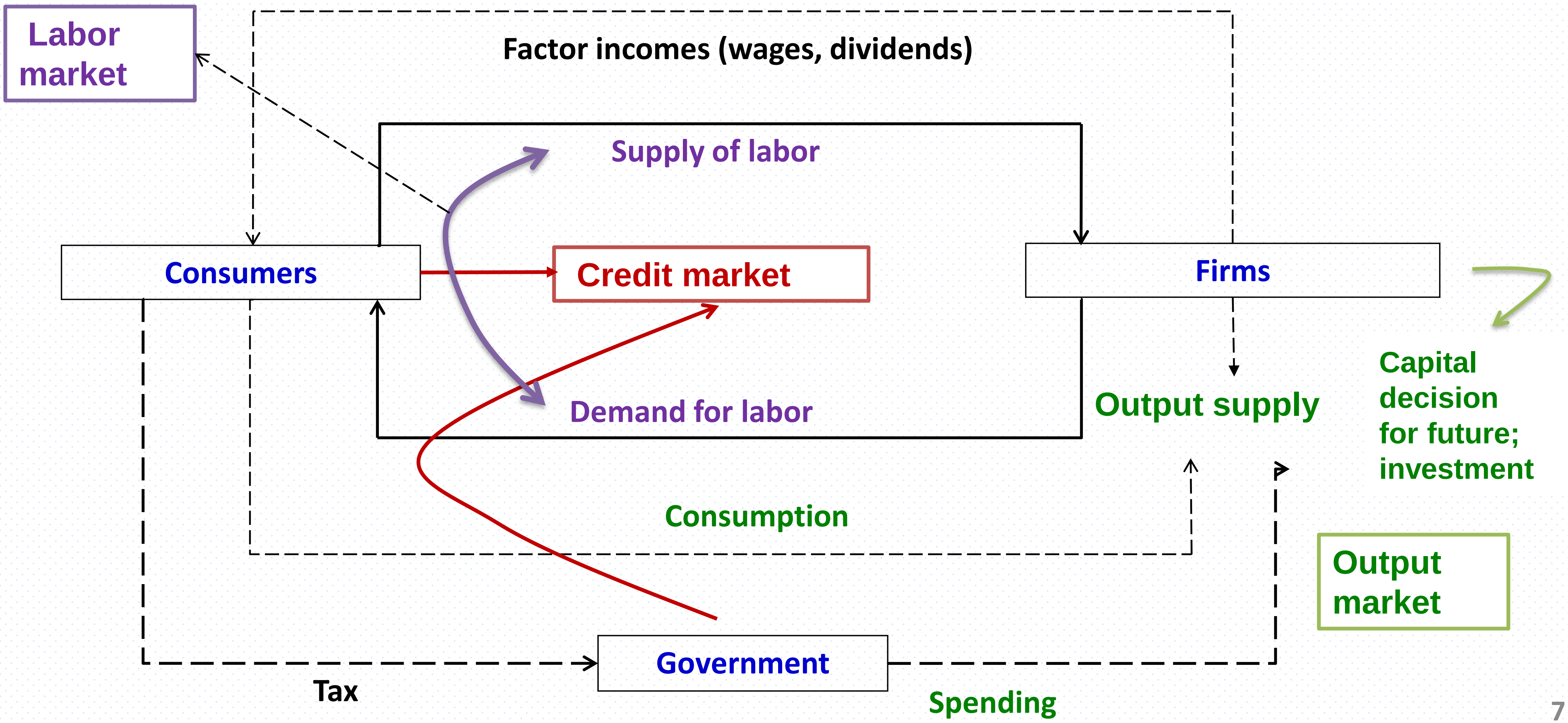
- The model assumes **two periods!**

- The **real model** (*no money*) with **three agents**
  - ***Representative consumer*** (consumption, labor supply and savings in each period)
  - ***Representative firm*** (production, labor demand and ***investment*** in each period)
  - **Government** (spending, taxes and borrowing)

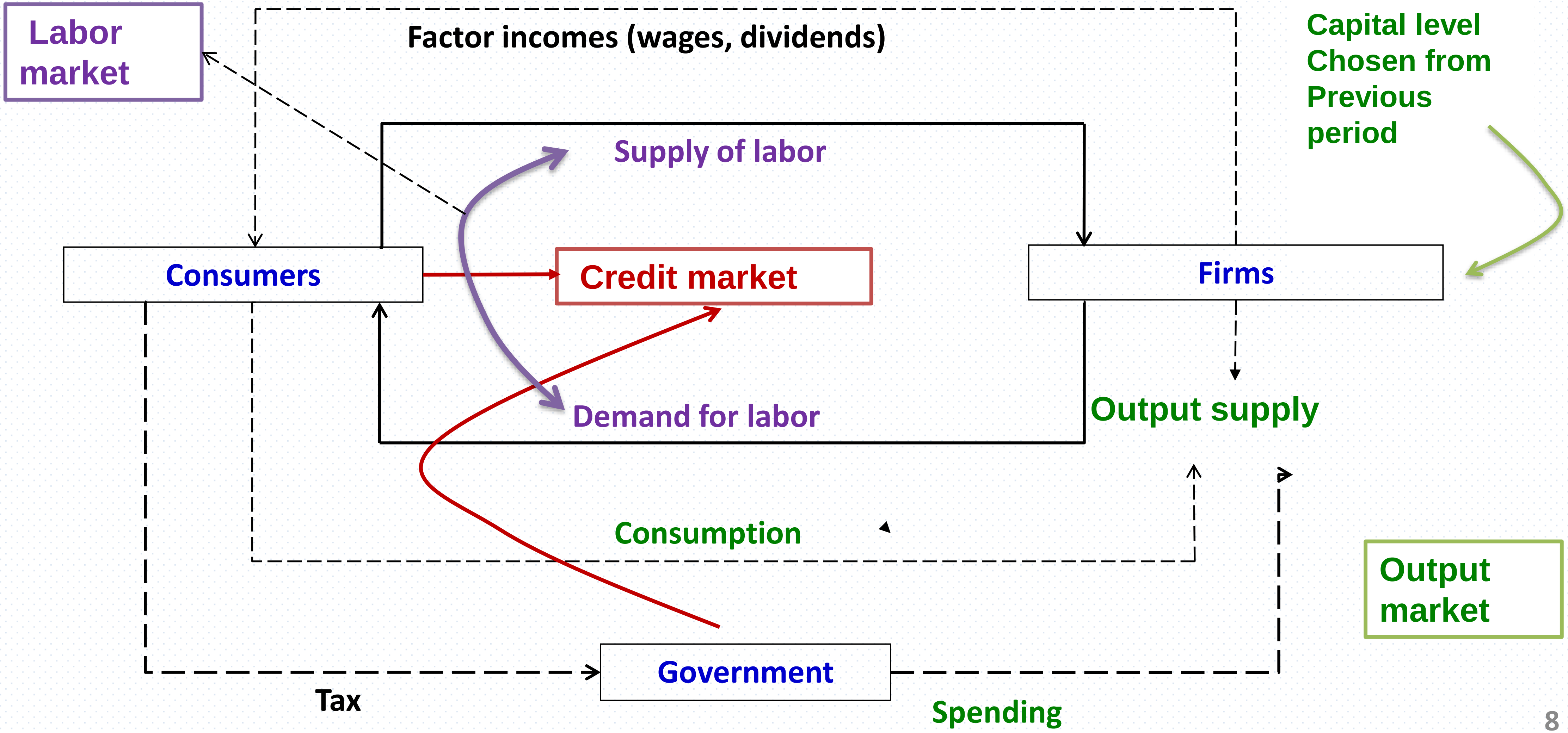
Structure of the real Intertemporal Model: markets

- **The labor market:** the firm's demand and the consumer's supply of labor
  - *The real wage rate*. (labor demand v.s. labor supply)
- **The output market:** the firm's supply and the consumer's demand for output
  - *The real interest rate. (output demand v.s. output supply)*
- **The credit market:** the supply for funds and demand for funds
  - *The real interest rate (saving v.s. borrowing)*

Circular flow of real intertemporal model: current



Circular flow of real intertemporal model: future



Roadmaps

- Optimizing-agent behavior (today)
- Equilibrium (next lecture: part II)
- Comparative static analysis (part III)

Consumer's optimal decisions

- **Work-leisure** in **current** and **future** periods.
- **Consumption-savings** in the current period.
  - h = total time available;
  - w and w' = current and future real wages;
  - r = the real interest rate;
  - T and T' = current and future lump-sum taxes;
  - C and C' = current and future consumptions;
  - L and L' = current and future leisure time;
  - Sp = private savings.

Current budget constraint

- The consumer is **a price-taker** (w, w', r and T are given).
  - w(h - L) = real-wage income;
  - π = dividend income from the firm;
  - T = lump-sum taxes paid to the government.

- Then, **disposable income** is:

$$C + S^p = w(h - l) + \pi - T$$

Future budget constraint

- The consumer still receives real wages, dividend income, and pays future taxes.
  - Also the principal and interest on savings.
  - No bequests; all wealth is consumed.

$$C' = w'(h - l') + \pi' - T' + (1 + r)S^p$$

Lifetime budget constraint

$$C + \frac{C'}{1+r} = w(h-l) + \pi - T + \frac{w'(h-l') + \pi' - T'}{1+r}$$

- The PV of **lifetime consumption** equals the PV of **lifetime disposable income**.

- Decision on the optimal bundles of C, C', L, and L' subject to the lifetime budget constraint.

Households Optimization problem

Households choose for optimal (C,L) and (C′,L′) that maximizes the utility function

$$U(C, L, C', L') = u(C, L) + u(C', L')$$

U = life-time intertemporal utility; u = periodic utility

subject to the budget constraint given by

$$C + \frac{C'}{1+r} = w(h-l) + \pi - T + \frac{w'(h-l') + \pi' - T'}{1+r}$$

The LaGrange method

- To solve for the household optimization problem, we use the LaGrange method.

Objective function: $z = f(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \dots, \boldsymbol{x_N})$
Constraint: $g(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \dots, \boldsymbol{x_N}) = 0$

Theorem: Solution to the above constrained optimization problem can be obtained by solving for the **critical point(s)** of the below **LaGrange function**

$$L = f(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \dots, \boldsymbol{x_N}) + \boldsymbol{\lambda} g(\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \dots, \boldsymbol{x_N})$$

The LaGrange method

- Suppose that λ is the LaGrange multiplier associated to the constraint set.

$$\mathcal{L} = \mathrm{u(C, L)} + \mathrm{u(C', L')} +$$

$$\lambda\left[w(h - L) + \pi - T + \frac{w'(h-L')+\pi'-T'}{1+r} - C - \frac{c'}{1+r}\right]$$

FOCs implied by the LaGrange method

$$\mathcal{L} = u(C,L) + u(C',L') + \lambda\left[w(h-L) + \pi - T + \frac{w'(h-L') + \pi' - T'}{1+r} - C - \frac{C'}{1+r}\right]$$

1. $[\mathrm{c}] \quad \rightarrow \mathrm{u}'_{\mathrm{c}}(\mathrm{C},\mathrm{L}) = \lambda$

2. $[\mathrm{c}'] \quad \rightarrow \mathrm{u}'_{\mathrm{c}'}(\mathrm{C}',\mathrm{L}') = \dfrac{\lambda}{1+r}$

3. $[\mathrm{L}] \quad \rightarrow \mathrm{u}'_{L}(\mathrm{C},\mathrm{L}) = \mathrm{w}\lambda$

4. $[\mathrm{L}'] \quad \rightarrow \mathrm{u}'_{L'}(\mathrm{C}',\mathrm{L}') = \dfrac{\mathrm{w}'\lambda}{1+r}$

Set of the optimal conditions

- From the 4 FOCs, we yield three important sets of optimality conditions:

  - Two (within-period) consumption-leisure trade-off

  - One intertemporal consumption-saving trade-off

  - One intertemporal current/future leisure trade-off

FOCs implied by the LaGrange method

1. $[c] \quad \rightarrow u'_c(C, L) = \lambda$

2. $[c'] \quad \rightarrow u'_{c'}(C', L') = \dfrac{\lambda}{1+r}$

3. $[L] \quad \rightarrow u'_L(C, L) = w\lambda$

4. $[L'] \quad \rightarrow u'_{L'}(C', L') = \dfrac{w'\lambda}{1+r}$

Current period (intratemporal) optimal condition

$$\frac{u'_L(C,L)}{u'_C(C,L)} = MRS_{L,C} = w$$

- The consumer chooses the optimal bundle of current leisure and consumption:
  - **The marginal rate of substitution of current leisure for current consumption** is equal to the real wage.
  - **w** is the relative price of leisure in terms of consumption goods.

Current period (intratemporal) optimal condition

$$\frac{u'_L(C,L)}{u'_C(C,L)} = MRS_{L,C} = w \Rightarrow u'_L(C,L) = w * u'_C(C,L)$$

- **Intuition:**
  - **Cost:** Taking less leisure 1 unit → utility drops by $\boldsymbol{u'_L(C,L)}$
  - **Benefit:** More wage earned by "w", and hence an increase in consumption by "w" units. → utility increases by $\boldsymbol{w * u'_c(C,L)}$

FOCs implied by the LaGrange method

1. [c] $\rightarrow u'_c(C, L) = \lambda$

2. [c′] $\rightarrow u'_{c'}(C', L') = \dfrac{\lambda}{1+r}$

3. [L] $\rightarrow u'_L(C, L) = w\lambda$

4. [L′] $\rightarrow u'_{L'}(C', L') = \dfrac{w'\lambda}{1+r}$

Future period (intratemporal) optimal condition

$$\frac{u'_{C'}(C',L')}{u'_{L'}(C',L')} = MRS_{L',C'} = w'$$

- The consumer chooses the optimal bundle of *future leisure* and *future consumption*:
  - **The marginal rate of substitution of future leisure for future consumption** is equal to the future real wage.
- The same intuition applies!

FOCs implied by the LaGrange method

1. [c] $\rightarrow u'_c(C, L) = \lambda$

2. [c′] $\rightarrow u'_{c'}(C', L') = \dfrac{\lambda}{1+r}$

3. [L] $\rightarrow u'_L(C, L) = w\lambda$

4. [L′] $\rightarrow u'_{L'}(C', L') = \dfrac{w'\lambda}{1+r}$

Intertemporal optimal condition

$$\frac{u'_C(C,L)}{u'_{C'}(C',L')} = MRS_{C,C'} = (1+r)$$

- The consumer chooses the optimal bundle of current and future consumption (savings):
  - **The marginal rate of substitution of current consumption for future consumption** is equal to the real interest rate.
  - **(1 + r)** is the relative price of current consumption in terms of future consumption

Intertemporal optimal condition

$$\frac{u'_C(C,L)}{u'_{c'}(C',L')} = (1+r) \Rightarrow \boldsymbol{u'_C(C,L) = (1+r)u'_{c'}(C',L')}$$

- **Intuition:**
  - **Cost:** Taking less consumption 1 unit → utility drops by $\boldsymbol{u'_C(\mathrm{C},L)}$
  - **Benefit:** More earning by (1+r) for future consumption → utility increases by $\boldsymbol{(1+r)u'_{c'}(C',L')}$

FOCs implied by the LaGrange method

1. [c] $\rightarrow u'_c(C, L) = \lambda$

2. [c′] $\rightarrow u'_{c'}(C', L') = \dfrac{\lambda}{1+r}$

3. [L] $\rightarrow u'_L(C, L) = w\lambda$

4. [L′] $\rightarrow u'_{L'}(C', L') = \dfrac{w'\lambda}{1+r}$

Intertemporal optimal condition

$$\frac{u'_L(C, L)}{u'_{L'}(C', L')} = MRS_{L,L'} = \frac{w(1+r)}{w'}$$

- The consumer chooses the optimal bundle of current and future leisure:
  - **The marginal rate of substitution of current leisure for future leisure** is equal to the real $\frac{w(1+r)}{w'}$.
  - $\frac{w(1+r)}{w'}$ is the relative price of current leisure in terms of future leisure.

Intertemporal optimal condition

$$\frac{u'_L(C,L)}{u'_{L'}(C',L')} = \frac{w(1+r)}{w'} \Rightarrow u'_L(C,L) = \frac{w(1+r)}{w'} u'_{L'}(C',L')$$

- **Intuition: (a bit complicated!)**
  - **Current Cost:** Taking less leisure 1 unit → utility drops by $u'_L(C,L)$
    - **Current benefit:** More earning by "w"; let's not use it up for now!
  - **Benefits realized in the future:**
    - **Option 1:** More earning by w*(1+r) for future consumption → utility increases by $w * (1+r) * u'_{C'}(C',L')$
    - **Option 2:** Let's not use for more future consumption, but instead for future leisure, the above is equivalent to $w * \frac{(1+r)}{w'} * u'_{L'}(C',L')$

Optimizing-agent behaviors

- From the four trade-off conditions (plus budget constraint), one can technically solve for optimal bundles and characterize the behavioral outcome of the optimizing-agent decision problem.

- These behavioral outcomes can be conceptualized by (i) **current labor supply behavior** and (ii) **current consumption demand behavior**

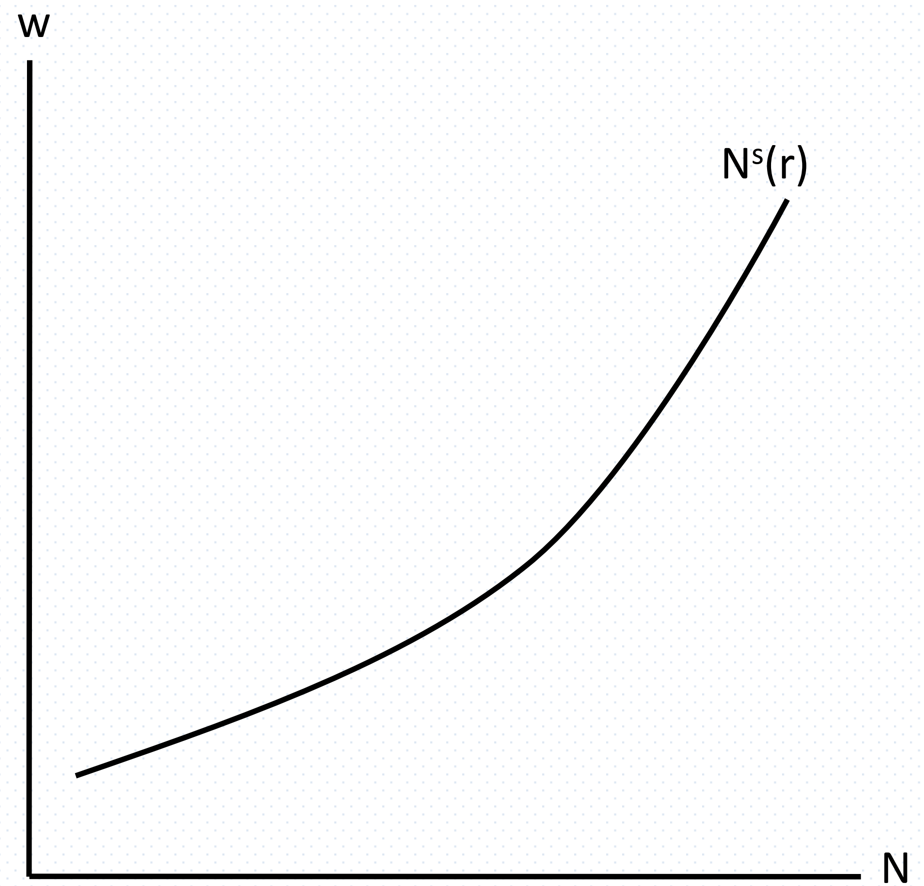
Current labor supply behavior

- The consumer provides labor supply to the firm through intertemporal/within-period decision.

- Factors which determine current labor supply:
  - **The current real wage;**
  - **The real interest rate;**
  - **Lifetime wealth.**

Current labor supply curve

- Current **labor supply increases with the real wage**, given r (assuming the dominant substitution effect).

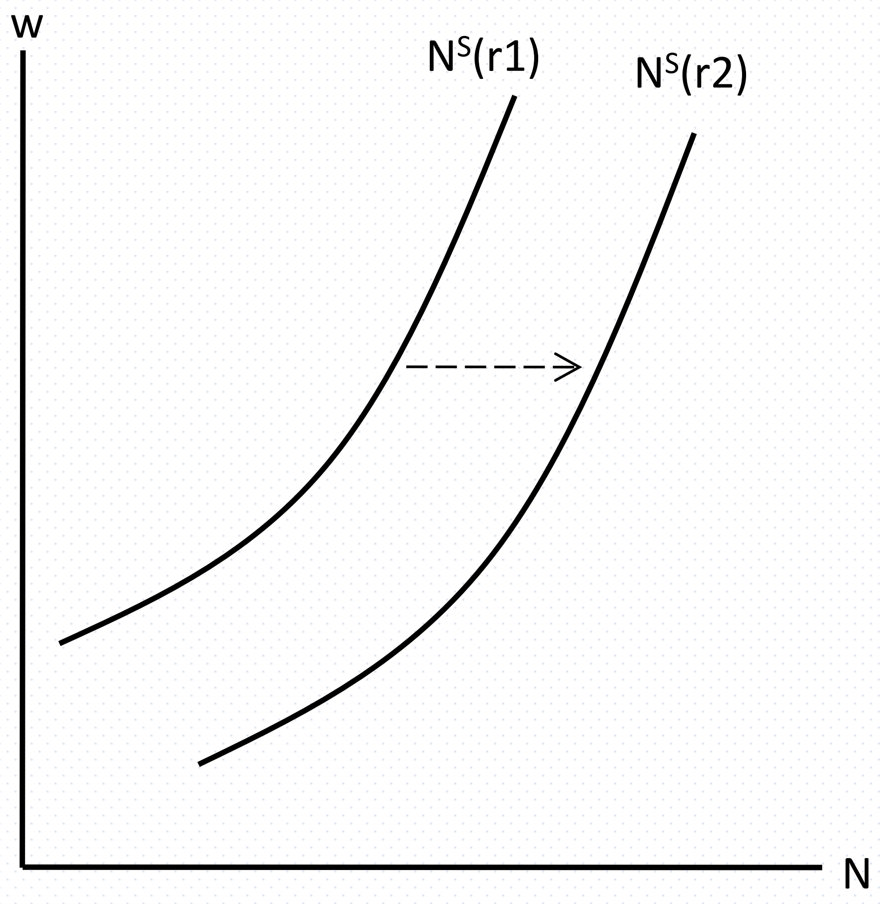


An increase in the real interest rate

- Given other things, current labor supply increases as the real interest rate increases.
  - **w(1+r)/w’** is the relative price of current leisure in terms of future leisure.
  - Given w and w’, a higher r means the higher price of current leisure in terms of future leisure.
  - Less current leisure, and more current supply of labor, assuming the dominant substitution effect.

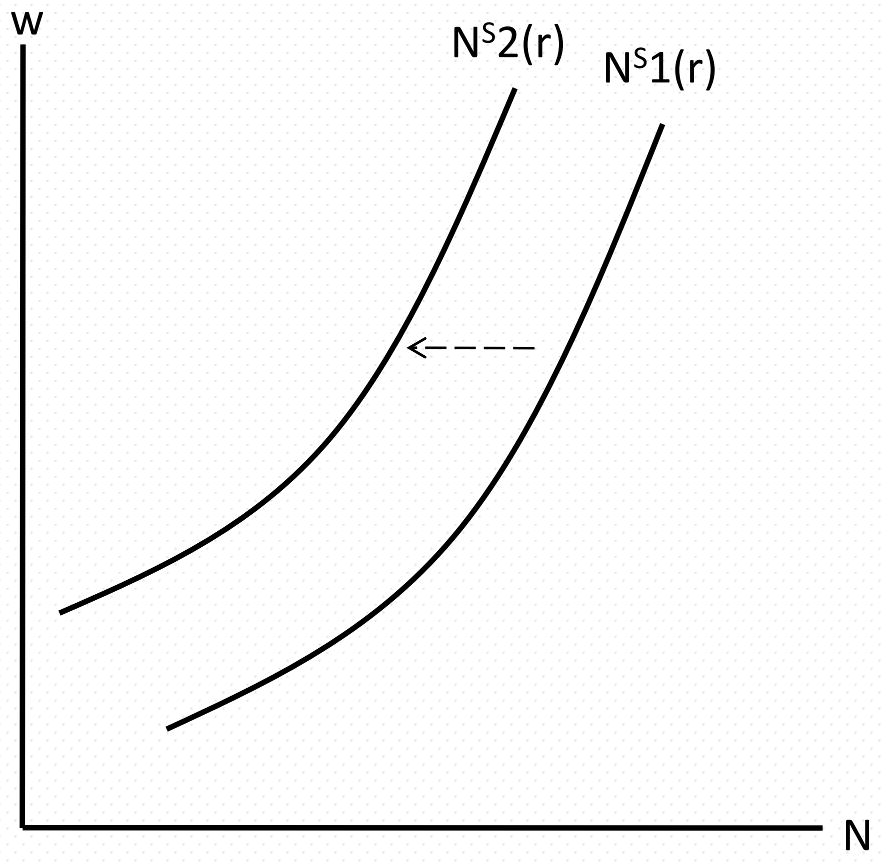
Labor supply *increases* with r

- Given w, **labor supply increases with the rising real interest rate** (r2 > r1), assuming the dominant substitution effect.



An increase in *lifetime wealth*

- Current leisure increases and **current labor supply decreases with rising lifetime wealth.**
  - Current and future consumptions also increase.



$$C+\frac{C'}{1+r}=\boxed{w(h-l)+\pi-T+\frac{w'(h-l')+\pi'-T'}{1+r}}$$

Demand for current consumption goods

- Consumption-saving decision results in demand for current consumption goods.

- There are three important factors the determine the behavior of current consumption goods
  - **Current income**
  - **Interest rate**
  - **Life-time wealth**

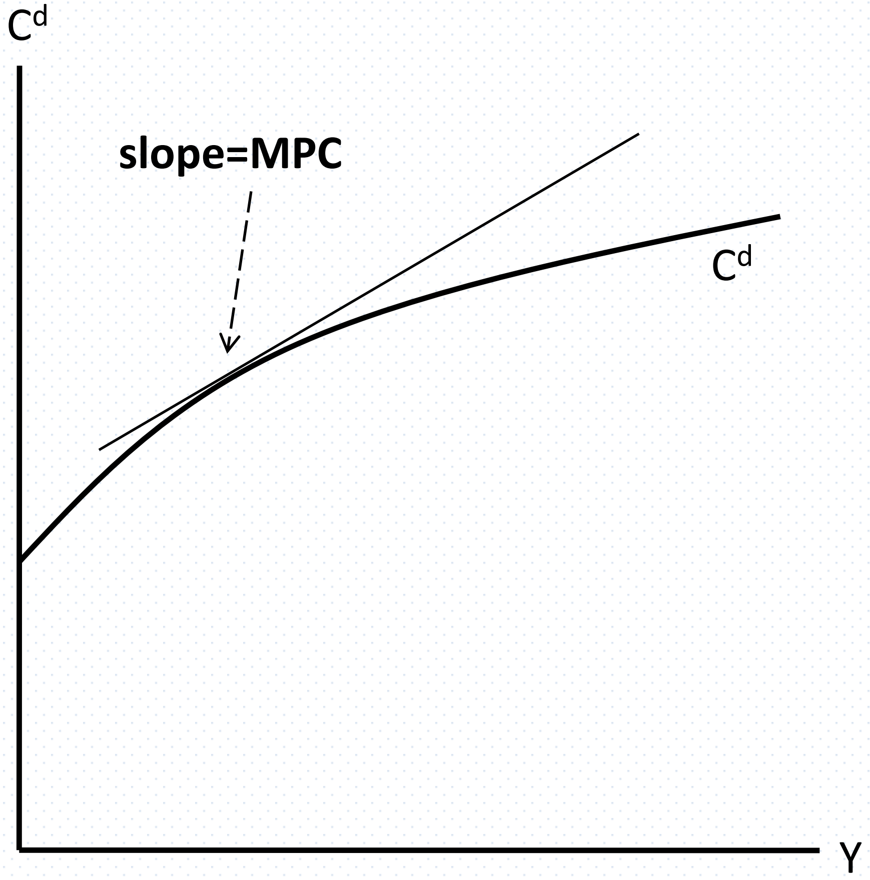
Demand for current consumption goods: current income

- **The individual demand for current consumption goods (C^d)** is a function of current income (Y), given r and we.
  - Current income (Y) = $w(h - L) + \pi - T$
  - The marginal propensity to consume (MPC) < 1 .

Demand for current consumption curve

$$C^d = f(Y, r, we)$$

$$MPC = \frac{\partial C^d}{\partial Y} < 1;$$



Demand for current consumption goods: real interest rate

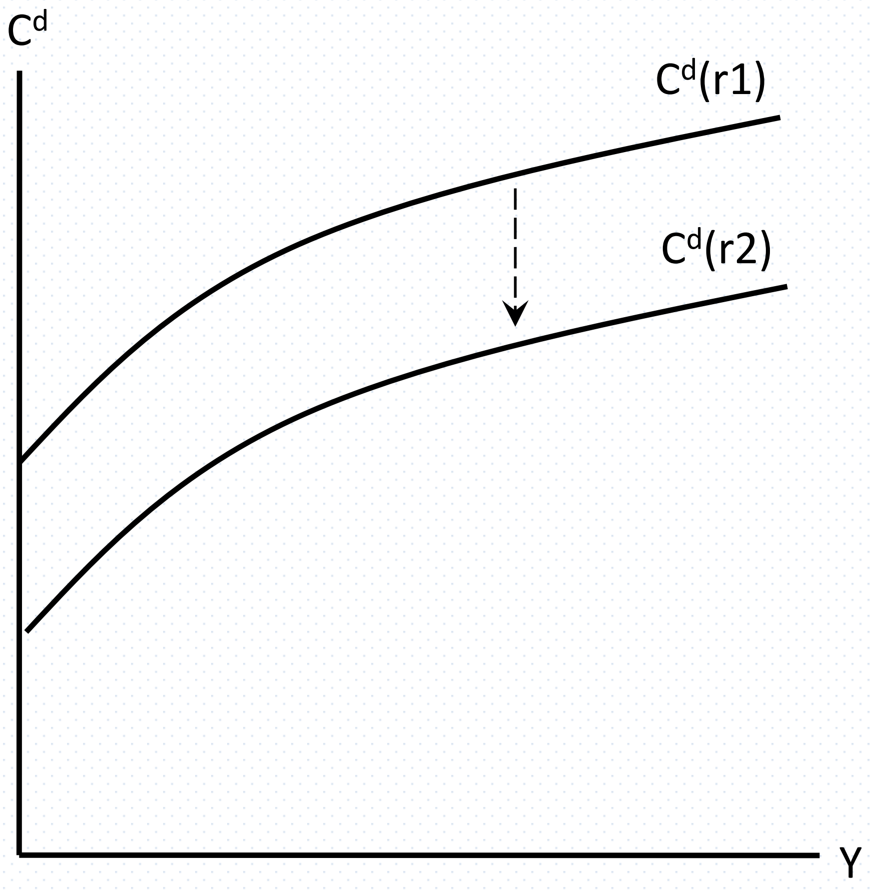
- A higher real interest rate (r) causes the demand to fall, assuming:
  - The substitution effect dominates the income effect; the consumer is a lender.
  - And other things fixed!

Demand for current consumption curve

$$C^d = f(Y, r, we)$$

$$\frac{\partial C^d}{\partial r} < 0;$$

- Given higher real interest rate (r2 > r1), the consumer reduces current consumption, (assuming stronger substitution effect and a lender.)



Demand for current consumption goods: life-time wealth

- The life-time wealth is given by

$$we = w(h - L) + \pi - T + \frac{w'(h - L') + \pi' - T'}{(1 + r)}$$

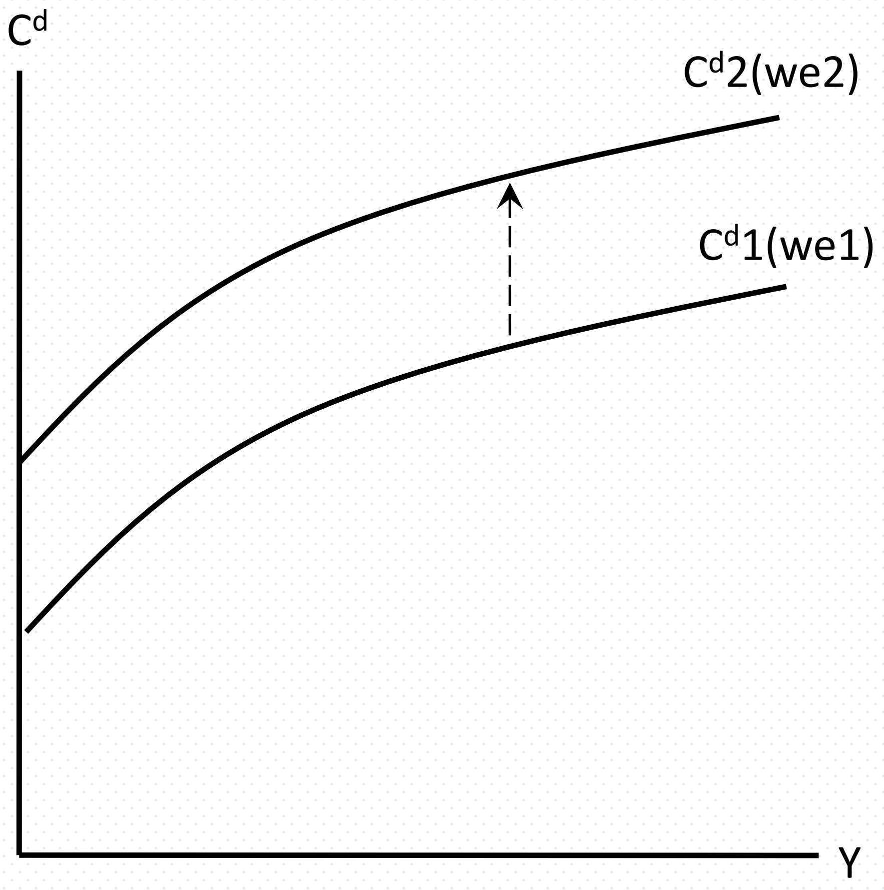
- Higher life-time wealth allows households to have more current consumption and leisure at the same time.
  - Assuming that both current consumption and leisure are normal goods.

Demand for current consumption

$$C^d = f(Y, r, we)$$

$$\frac{\partial C^d}{\partial we} > 0$$

- With we2>we1, an increase in ***lifetime wealth*** raises current consumption.



Representative firm

- In this model, firm operates for **two periods**, using their **own production technology**

- It aims at maximizing **firm's value (V)** as given by the present value of life-time firm's profit, i.e. **.** $\pi + \frac{\pi'}{1+r}$

Production technology

$$Y = zF(K, N)$$

- Y = current output;
- z = current total factor productivity;
- K = current capital stock;
- N = current labor input.
- And the **future production function**:

$$Y = z'F(K', N')$$

K → K': Investment

- **Future capital stock is current capital stock net of depreciation plus investment.**

$$K' = (1-d)K + I$$

  - d = the rate of depreciation;
  - I = current investment.

- **Why firm invest?**
  - The firm's investment is foregone current profits (consumption) for future profits; increase future capacity
  - Overall firm's value can increase given optimal investment plan

Firm's problem

- Given the **production technology** owned, firm chooses for (i) **optimal labor (N, N') in each period** and (ii) **optimal *investment (I)*** so as to maximize the life-time profits.

$$V = \pi + \frac{\pi'}{1+r}$$

$$\pi' = Y' - w'N' + (1-d)K'$$

$$\pi = Y - wN - I$$

The leftover capital stock in the future period can be sold off as junk value.

(1 - d)K' = capital stock remaining as junk at the end of the future period.

Optimality conditions

$$V = \pi + \frac{\pi'}{1+r}$$

$$= zF(K,N) - wN - I + \frac{z'F(K',N') - wN' + (1-d)K'}{1+r}$$

- [N]: $zF'_N(K,N) = w$
- [N']: $z'F'_N(K',N') = w'$
- [I]: $1 = \frac{z'F'_{K'}(K',N') + (1-d)}{1+r}$

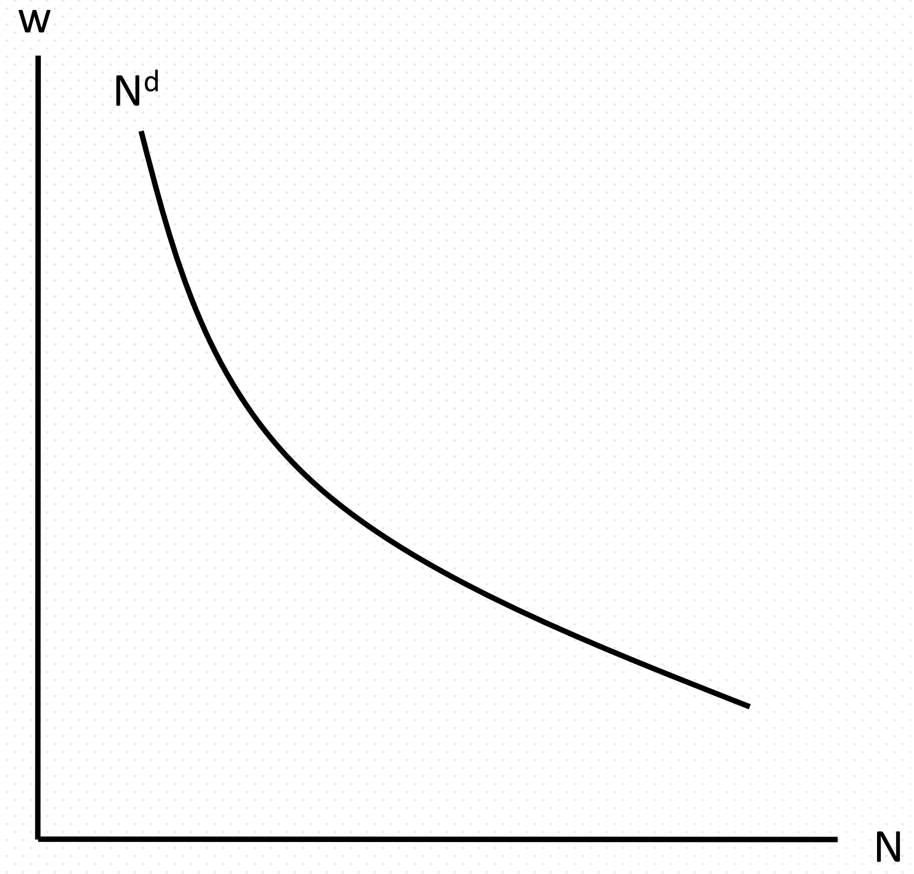
Optimal labor input

- [N]: $zF'_N(K,N) = w$
  - The firm hires current labor until the current marginal product of labor equals the current real wage ($MP_N = zF'_N(K,N) = w$).

- Thus the firm's MP_N curve is also **the firm's current labor demand curve**.
  - An increase in current z or K raises MP_N and current labor demand.

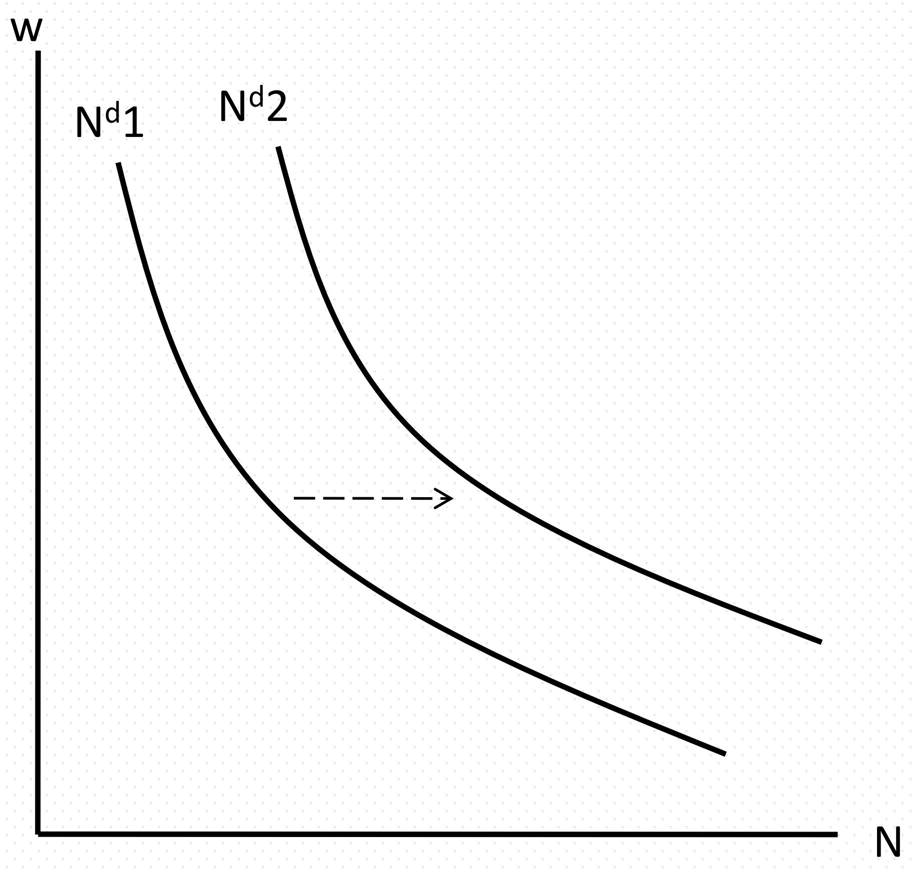
Current labor demand

- The current labor demand: $MP_N = w$.
- MP_N is falling as the labor input increases.



Labor demand with rising z or K

- An increase in current z or K shifts the current labor demand curve to the right.



Optimal investment decision

- [I]: $$1 = \frac{z' F'_{K'}(K', N') + (1-d)}{1+r}$$
  - The firm invests to the point where the marginal benefit from investment equals marginal cost.

- **MC(I) = marginal cost of investment** = PV of profits (V) given up for one unit of capital.
  - One unit of investment reduces current π (and V) by one unit; **MC(I) = 1**

Optimal investment decision

- [I]: $1 = \frac{z' F'_{K'}(K', N') + (1 - d)}{1 + r}$
- **MB(I) = marginal benefit of investment** = additional units of V (PV of profits) received from **one extra unit of current investment.**
  - **Benefit 1:** $z' F'_{K'}(K', N') =$ MP$_K$' = additional output from one extra unit of K'.
  - **Benefit 2:** Quantity of capital left from depreciation at the end of the future period (1 - d) for liquidation.
- Additional future profits = $z' F'_{K'}(K', N') + (1 - d)$
- PV of additional benefits = $\frac{z' F'_{K'}(K', N') + (1 - d)}{1 + r}$

Optimal investment decision

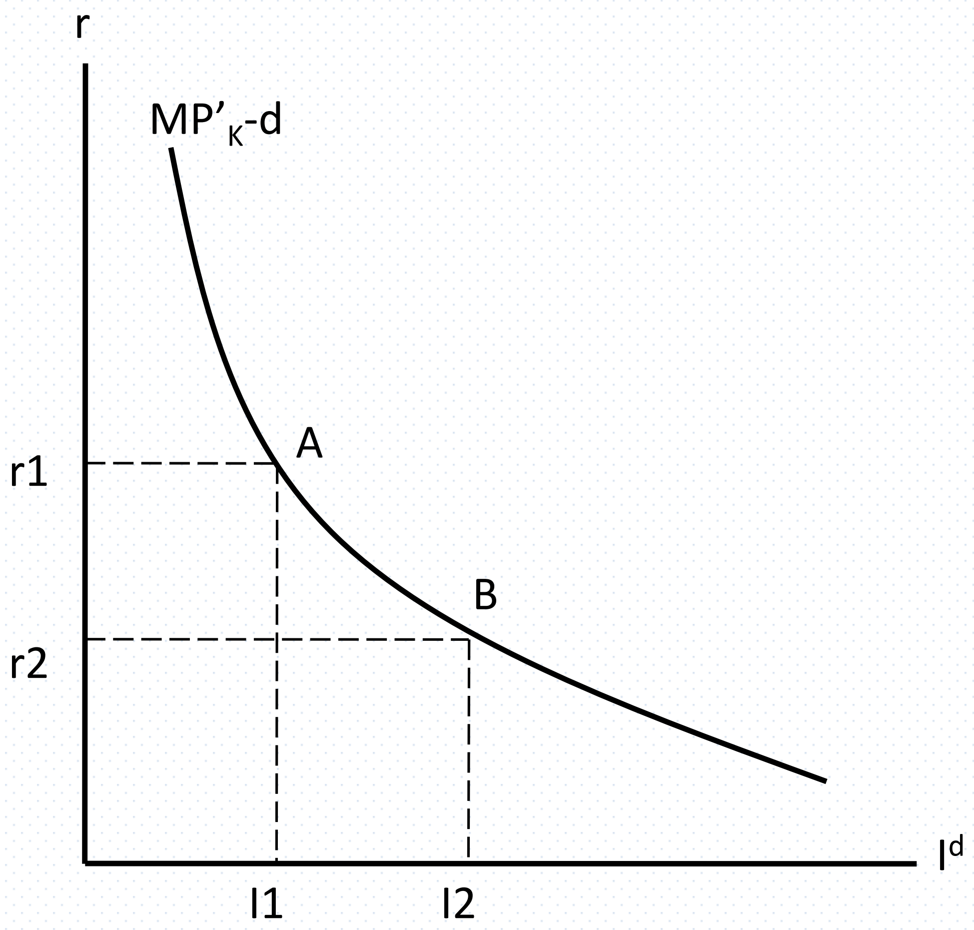
$$1 = \frac{z' F'_{K'}(K', N') + (1 - d)}{1 + r} \Rightarrow \boldsymbol{MP_{K'} + (1 - d) = 1 + r}$$

- The firm chooses for the optimal scale of physical investment where return on physical investment equals to return on financial investment, i.e. opportunity cost of project funding
  - **r = the opportunity cost of more capital** = the rate of return on the alternative asset (bonds) otherwise earned by the consumer who owns the firm.

Optimal investment curve

- With lower "r", firm chooses for higher amount of future physical stock (K'), and hence the optimal scale of investment (I^d)

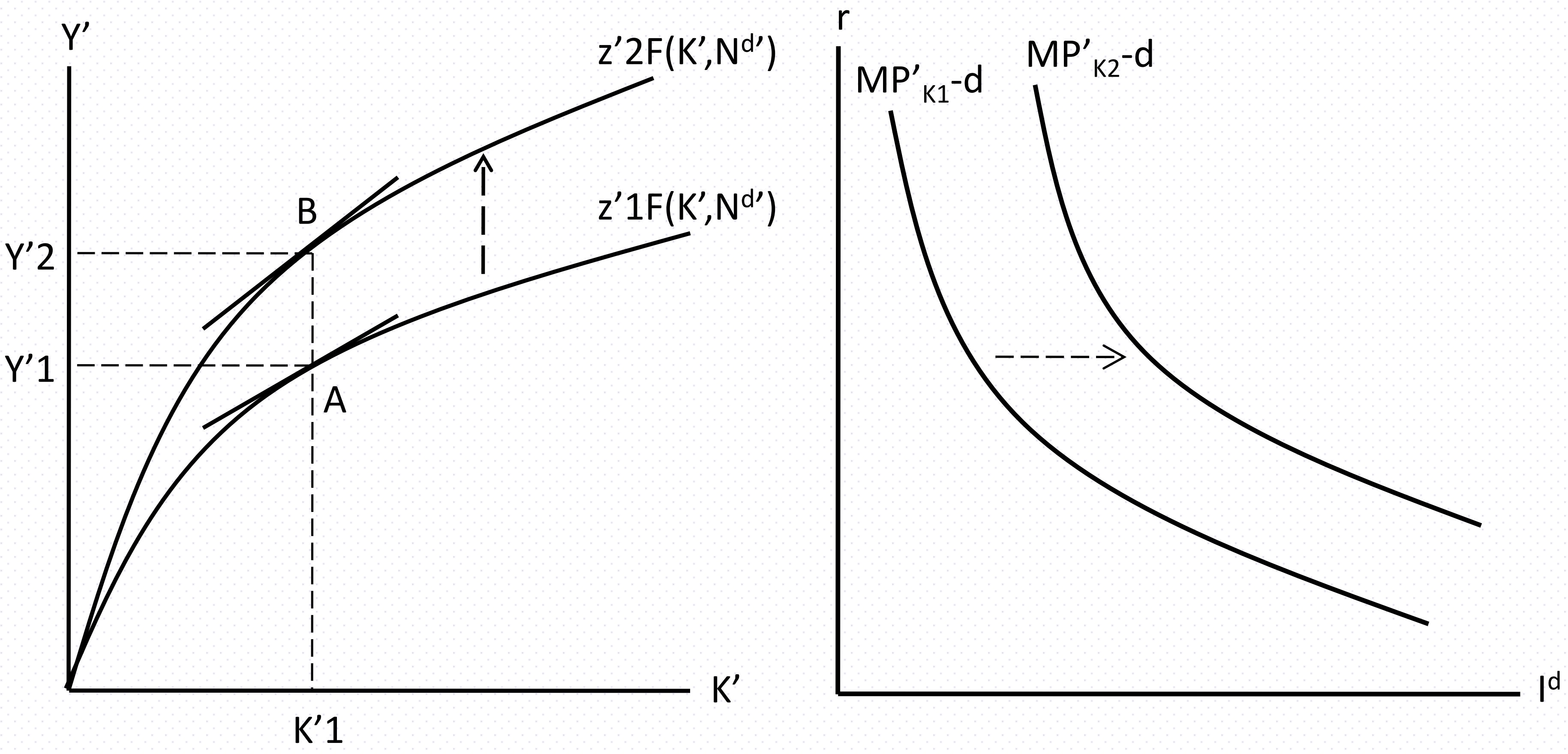
- Investment demand is downward sloping in "r", given other factors



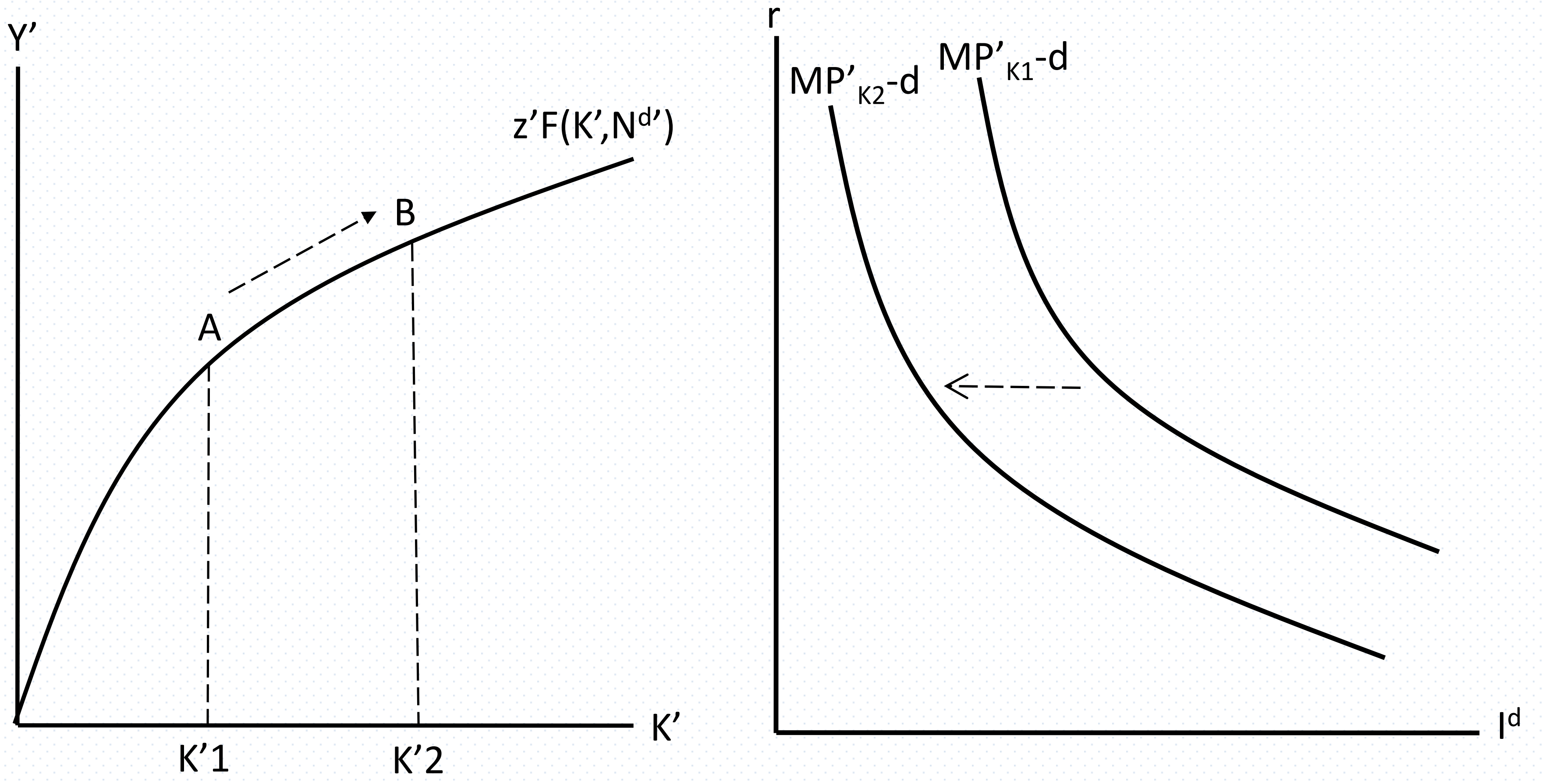
Changes in z' and K

- Factors affecting future marginal product of capital shift the optimal investment curve.
- Higher **future total factor productivity** (z') increases **future MP'_K** and current optimal investment.
  - The optimal investment curve shifts to the right.
- Higher **current capital stock** results in larger future net capital stock and lower MP'_K.
  - The optimal investment curve shift to the left.

Higher z' and MP'_K



Higher current K and lower MP'_K

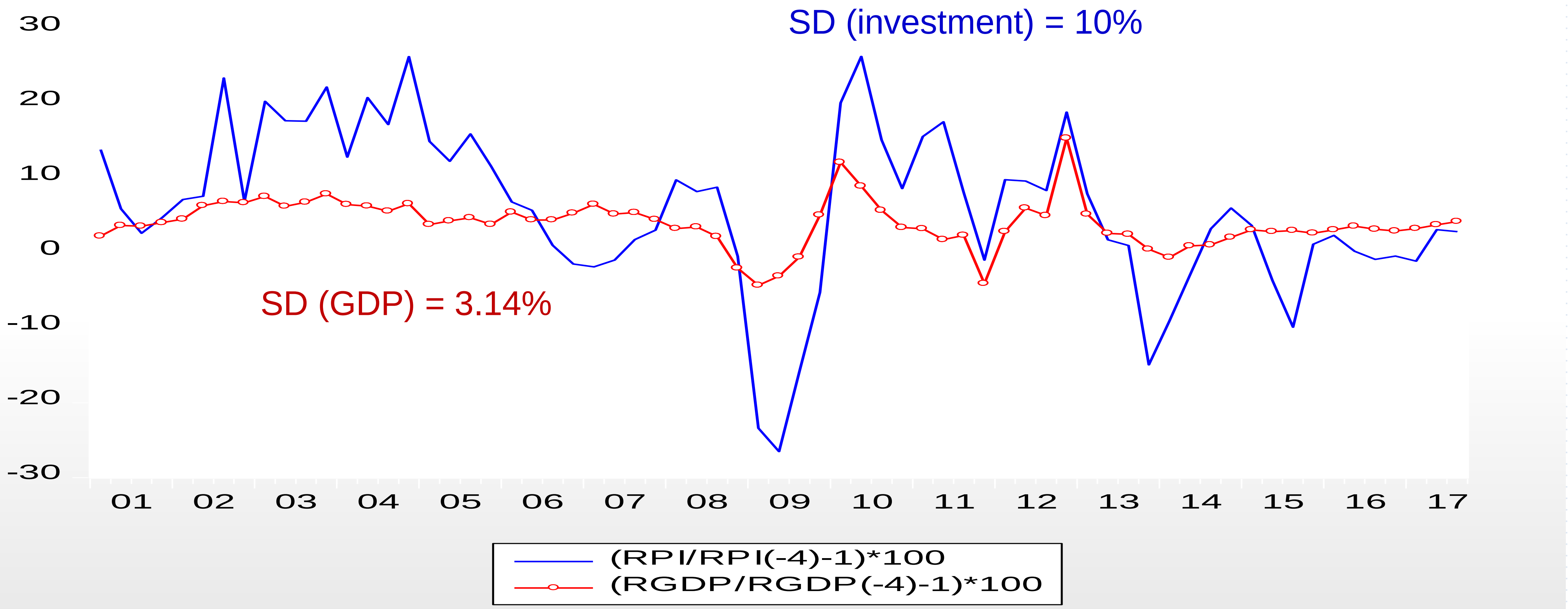


Empirical application: Volatile investment and GDP

- Aggregate consumption is less variable than income due to consumption smoothing.

- Investment is **much more volatile** --- short-run economic fluctuations.
  - Investment responds to perceived marginal rates of return to investment.
  - Changes in the real interest rate cause movements along the investment curve.
  - Changes in future total factor productivity shift the investment curve.

Empirical application: Volatile investment and GDP



Government sector

- Government purchases of consumption goods (G and G’) are exogenously determined.
- Government financing:
  - Current lump-sum taxes and bond sale;
  - Future lump-sum taxes and payments of the principal and interest.

$$G + \frac{G'}{1+r} = T + \frac{T'}{1+r}$$

Competitive equilibrium

- **The labor market**:
  - The consumer supplies labor service.
  - The firm demands labor service.
  - The real wage and the level of employment.
- **The output market**:
  - The consumer, the firm and government purchase output.
  - The firm supplies the goods.
  - The real interest rate and the level of aggregate output.