Introduction to Binomial Trees I

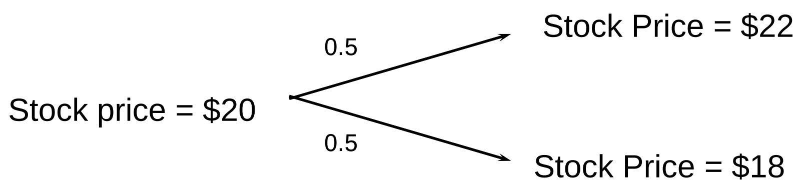
Chapter 11

Outline

- Binomial tree
- Setting up a risk free portfolio with options and the underlying
- Setting up a portfolio that replicates the option payoff
- Risk neutral pricing & the irrelevance of expected return
- Generalizing the pricing equation

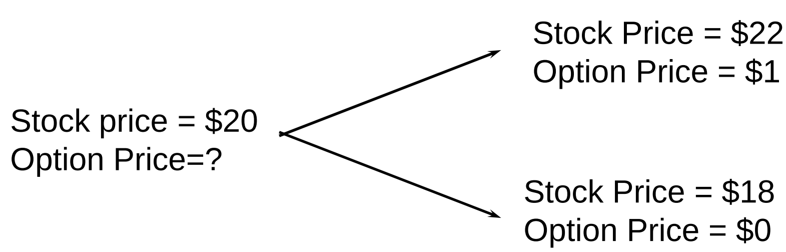
A Simple Binomial Model

- A stock price is currently $20
- In three months it will be either $22 or $18



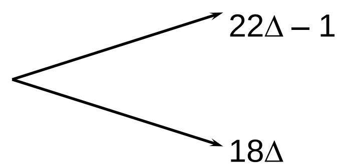
A call option

A 3-month call option on the stock has a strike price of 21.



Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ shares
  short 1 call option



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio

- The riskless portfolio is:
  long 0.25 shares
  short 1 call option
- The value of the portfolio in 3 months at the up state is $22 \times 0.25 - 1 = 4.50$
- The value of the at the low state is 18*0.25 = 4.50

Setting Up a Riskless Portfolio

- Consider the Portfolio: long 0.25 shares short 1 call option

$22*0.25 - 1 = 4.50$

$18*0.25 = 4.50$

- Portfolio is now riskless, so we can discount the certain cash flows by the risk free rate
- The value of the portfolio today is $4.5e^{-0.12\times0.25} = 4.3670$

Valuing the Option

- The portfolio that is

  long 0.25 shares
  short 1 option

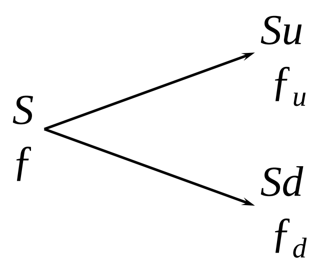
  is worth 4.367
- The value of the portfolio today

  shares is $4.367 = \Delta S - c = 0.25*20 - c$
- The value of the option is therefore

  c= 0.633 (= 5.000 – 4.367)

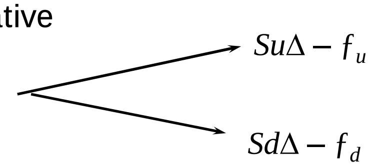
Generalization

A derivative lasts for time *T* and is dependent on a stock



Generalization

- Consider the portfolio that is long Δ shares and short 1 derivative



- The portfolio is riskless when $Su\Delta - f_u = Sd\,\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{Su - Sd}$$

Generalization

- Value of the portfolio at time T is $Su\Delta - f_u$
- Value of the portfolio today is $(Su\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S\Delta - f$
- Hence $$f = S\Delta - (Su\Delta - f_u)e^{-rT}$$

Generalization

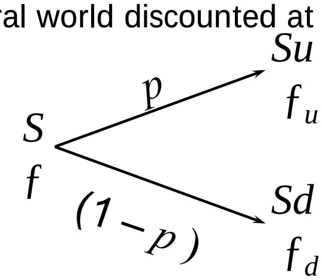
- Substituting for Δ we obtain

$$f = [\,p f_u + (1 - p)f_d\,]e^{-rT}$$

where $$p = \frac{e^{rT} - d}{u - d}$$

Risk-Neutral Valuation

- $f = [\, p\, f_u + (1 - p\,)f_d\,]e^{-rT}$
- The variables p and $(1 - p\,)$ can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



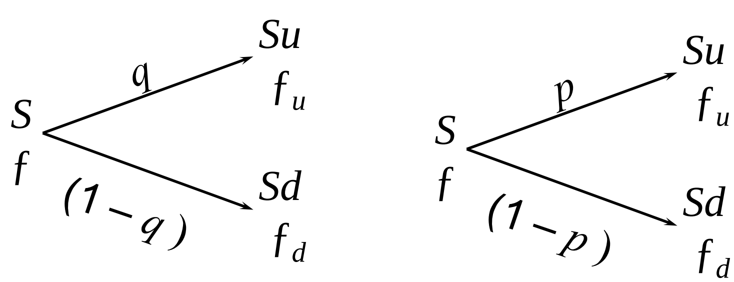
Irrelevance of Stock's Expected Return

- The probabilities of the underlying stock may be q and (1-q)
- The current stock price S is the expected value of future cash flow discounted at the rate which considers the uncertainty of the cash flow
- When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant
- The risk of the underlying stock is priced in the current stock price

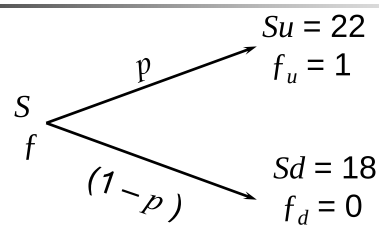
Irrelevance of Stock's Expected Return

- The option valuation is a relative pricing technique
- Because the risk of the underlying asset is included in its current price, we can price the option as if we were in the 'risk neutral' world
- In the risk neutral world all asset's cash flow are discount using the risk free rate

Risk-Neutral Valuation



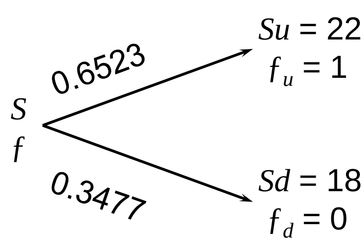
Original Example Revisited



- Since p is a risk-neutral probability
  $20 = e^{-0.12 \times 0.25} [22p + 18(1 - p)]; p = 0.6523$
- Alternatively, we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option



The value of the option is

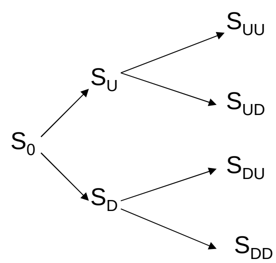
$e^{-0.12 \times 0.25} [0.6523 \times 1 + 0.3477 \times 0]$

$= 0.633$

Multi-stage Binomial Option Pricing

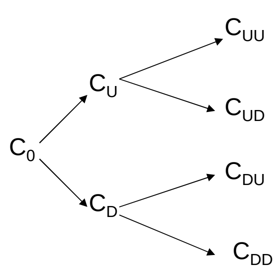
Just solve series of one-period models. Work backwards!

Step 1: Work out all stock prices. In two-period model, there are 2+4 = 6 of them.



Multi-stage Binomial Option Pricing

Step 2: Work out all C_{XX} at the termination (4 of them). Just solve $C_{XX}= \max(0,S_{XX}-X)$



Step 3: Use one-period model to solve for C_X (2 of them)

Step 4: Use one-period model to solve for C_0 .

Option Pricing: Two & n Subperiods

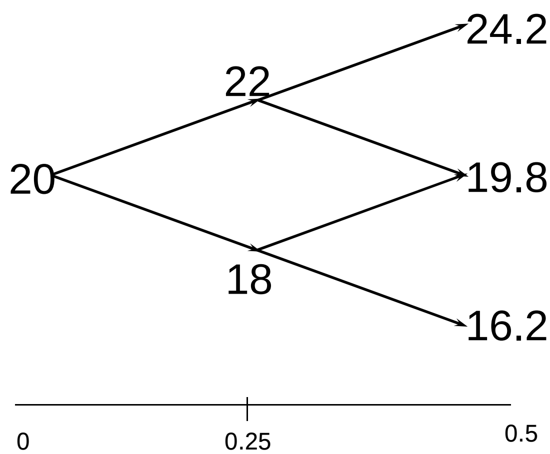
$$C_u = e^{-rT/2}\left[p \cdot C_{uu} + (1-p) \cdot C_{ud}\right]$$
$$C_d = e^{-rT/2}\left[p \cdot C_{ud} + (1-p) \cdot C_{dd}\right]$$

$$C = e^{-rT}\left[p^2 \cdot C_{uu} + 2 \cdot p \cdot (1-p) \cdot C_{ud} + (1-p)^2 \cdot C_{dd}\right]$$

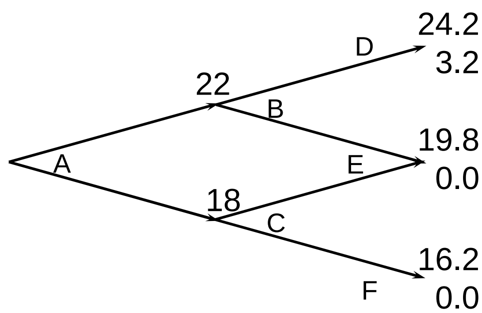
For n subperiods

$$C_{u..n} = e^{-rT/n}\left[p \cdot C_{uu..n} + (1-p) \cdot C_{ud..n}\right]$$
$$C_{d..n} = e^{-rT/n}\left[p \cdot C_{ud..n} + (1-p) \cdot C_{dd..n}\right]$$

A six months option with K=21



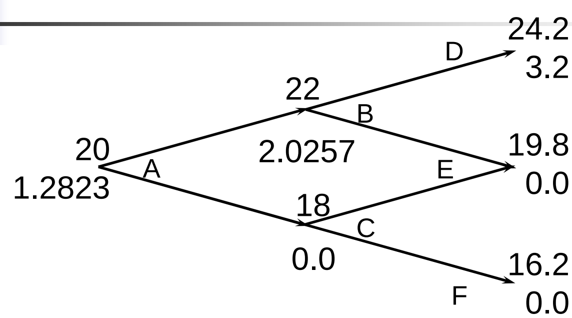
Valuing a Call Option



p for this binomial tree is

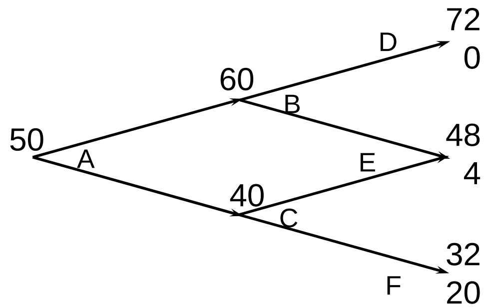
$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing a Call Option



- Value at node C
  $= e^{-0.12 \times 0.25}(0.6523 \times 0 + 0.3477 \times 0) = 0$
- Value at node B
  $= e^{-0.12 \times 0.25}(0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$
- Value at node A
  $= e^{-0.12 \times 0.25}(0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823$

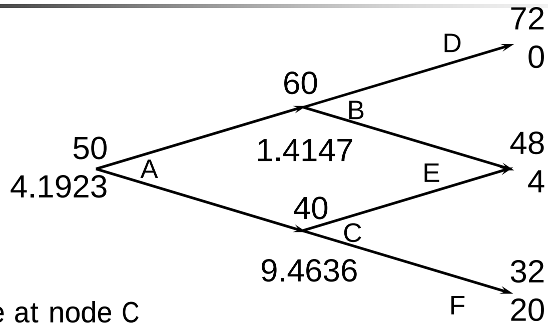
A Put Option Example; K=52; T=2; r=5%



$$u = 60/50 = 1.2;\ d = 40/50 = 0.8$$

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05\times 1} - 0.8}{1.2 - 0.8} = 0.6282$$

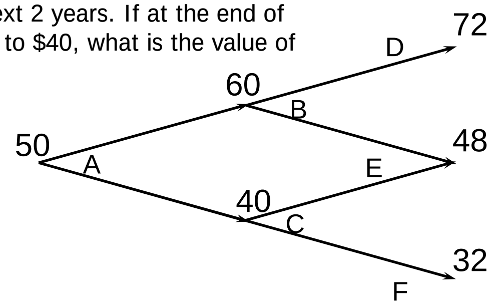
A Put Option Example; K=52



- Value at node C
  $= e^{-0.05\times 1}(0.6282\times 4 + 0.3718\times 20) = 9.4636$
- Value at node B
  $= e^{-0.05\times 1}(0.6282\times 0 + 0.3718\times 4) = 1.4147$
- Value at node A
  $= e^{-0.05\times 1}(0.6282\times 1.4147 + 0.3718\times 9.4636) = 4.1923$

Concept check

Consider a put option with K=55 and T=2 years. Interest rate is at 5%. This is the tree of the underlying stock for the next 2 years. If at the end of year 1 the stock price falls to $40, what is the value of option?

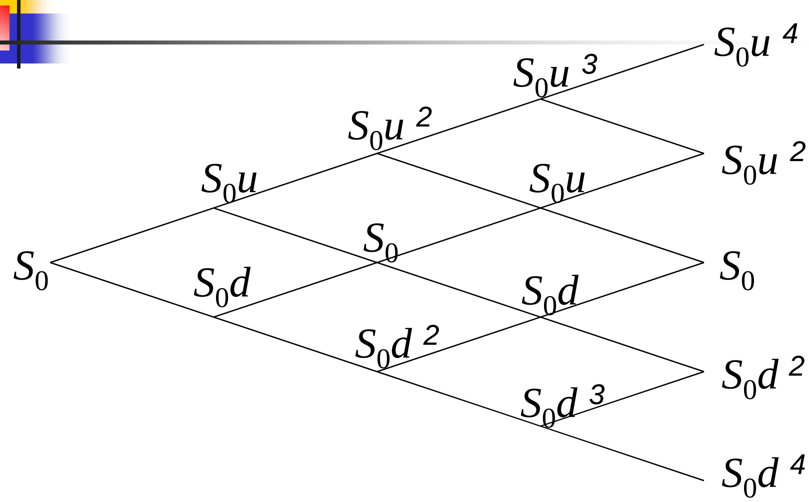


G) 12.31

Y) 9.46

R) 10.11

Four period tree



Summary

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