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CHAPTER 1

SPEAKING MATHEMATICALLY

1.1 Variables

A variable is sometimes thought of as a mathematical “John Doe” because you can use it as a placeholder when you want to talk about something but either (1) you imagine that it has one or more values but you don’t know what they are, or (2) you want whatever you say about it to be equally true for all elements in a given set, and so you don’t want to be restricted to considering only a particular, concrete value for it.

Example 1.1.1
Writing Sentences Using Variables

Use variables to rewrite the following sentences more formally.

a. Are there numbers with the property that the sum of their squares equals the square of their sum?

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Example 1.1.1
Writing Sentences Using Variables

b. Given any real number, its square is nonnegative.

Some Important Kinds of Mathematical Statement

Three of the most important kinds of sentences in mathematics are universal statements, conditional statements, and existential statements:

A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero.*)

A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6.*)

Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even.*)

Universal Conditional Statement

Universal statements contain some variation of the words "for all" and conditional statements contain versions of the words "if-then." A ***universal conditional statement*** is a statement that is both universal and conditional. Here is an example

For all animals a, if a is a dog, then a is a mammal.

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Example 1.1.2 Rewriting a Universal Conditional Statement

Fill in the blanks to rewrite the following statement:

For all real numbers x, if x is nonzero then x^2 is positive.

a. If a real number is nonzero, then its square_____.

b. For all nonzero real numbers x,_______________.

Example 1.1.2 Rewriting a Universal Conditional Statement

c. If x_________________,then______________.

d. The square of any nonzero real number is_____.

e. All nonzero real numbers have_______________.

Universal Existential Statement

A ***universal existential statement*** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something.

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Example 1.3.3 Rewriting a Universal Existential Statement

Fill in the blanks to rewrite the following statement: Every pot has a lid.

a. All pots_____________.

b. For all pots P, there is____________.

c. For all pots P, there is a lid L such that_________.

Existential Universal Statements

An ***existential universal statement*** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind. For example:

There is a positive integer that is less than or equal to every positive integer.

Example 1.1.4 Rewriting an Existential Universal Statement

Fill in the blanks to rewrite the following statement in three different ways:

There is a person in my class who is at least as old as every person in my class.

a. Some ______________ is at least as old as __________.

b. There is a person p in my class such that p is ______________________.

Example 1.1.4 Rewriting an Existential Universal Statement

c. There is a person p in my class with the property that for every person q in my class, p is__________________.

1.2 The Language of Sets

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor(1845-1918). For most mathematical purpose we can think of a set intuitively as Cantor did, simply as a collection of elements. For instance, if C, and if I is the set of all integers from 1 to 100, then the number 57 is an element of I.

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1.2 The Language of Sets

- **Notation**

If S is a set, the notation $x \in S$ means that x is an element of S. The notation $x \notin S$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, {1, 2, 3} denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write {1, 2, 3, . . . , 100} to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write {1, 2, 3, . . .} to refer to the set of all positive integers. (The symbol . . . is called an **ellipsis** and is read "and so forth.")

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Example 1.2.1
Using the Set-Roster Notation

a. Let $A = \{1,2,3\}, B = \{3,1,2\}$,and $C = \{1,1,2,3,3,3\}$. What are the elements of A, B, and C? How are A, B, and C related?

Example 1.2.1

Using the Set-Roster Notation

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

Example 1.2.1
Using the Set-Roster Notation

d. For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Certain sets of numbers are so frequently referred to that they are given special symbolic names.

Symbol	Set
R	set of all real numbers
Z	set of all integers
Q	set of all rational numbers, or quotients of integers

• Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all ($\{$) — such that ($\mid$)

Example 1.2.2
Using the Set-Builder Notation

Given that R denotes the set of all real numbers, Z denotes the set of all integers, and Z^+ the set of all positive integers, describe each of the following sets.

a. $\{x \in R \mid -2 < x < 5\}$

Example 1.2.2
Using the Set-Builder Notation

b. $\{x \in Z | -2 < x < 5\}$

c. $\{x \in Z^{+} | -2 < x < 5\}$

Subsets

• Definition

If A and B are sets, then A is called a **subset** of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B.

Symbolically:

$A \subseteq B$ means that For all elements x, if $x \in A$ then $x \in B$.

The phrases *A is contained in B* and *B contains A* are alternative ways of saying that A is a subset of B.

It follows from the definition of subset that for a set A not to be a subset of a set B means that there is at least one element of A that is not an element of B. Symbolically:

$A \nsubseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

• Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

Example 1.2.3 Subsets

Let $A = Z^{+}, B = \{n \in Z | 0 \leq n \leq 100\}$, and $C = \{100,200,300,400,500\}$. Evaluate the truth and falsity of each of the following statements.

a. $B \subseteq A$

Example 1.2.3 Subsets

b. C is a proper subset of A

c. C and B have at least one element in common

Example 1.2.3 Subsets

d. $C \subseteq B$

e. $C \subseteq C$

Example 1.2.4

Distinction between $\in$ and $\subseteq$

Which of the following are true statements?

a. $2 \in \{1,2,3\}$ b. $\{2\} \in \{1,2,3\}$ c. $2 \subseteq \{1,2,3\}$

d. $\{2\} \subseteq \{1,2,3\}$ e. $\{2\} \subseteq \{\{1\},\{2\}\}$ f. $\{2\} \in \{\{1\},\{2\}\}$

Cartesian Products

• Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 1.2.5 Ordered Pairs

a. Is $(1,2) = (2,1)$?

Example 1.2.5 Ordered Pairs

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of (1,1)?

• Definition

Given sets A and B, the **Cartesian product of A and B**, denoted $A \times B$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 1.2.6 Cartesian Products

Let $A = \{1,2,3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $\mathrm{B} \times A$

Example 1.2.6 Cartesian Products

c. Find $\mathrm{B} \times B$

d. How many elements are in $\mathrm{A} \times B$, $\mathrm{B} \times A$, and $\mathrm{B} \times B$?

Example 1.2.6 Cartesian Products

e. Let R denote the set of all real numbers.
Describe $R \times R$.

1.3 The Language of Relations and Functions

The objects of mathematics may be related in various ways. A set A may be said to be related to a set B if A is a subset of B,or if A is not a subset of B, o if A and B have at least one element in common. A number x may be said to be related to a number y if $x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$. Two identifiers in a computer program may be said to be related if they have the same first eight characters, or if the same memory location is used to store their values when the program is executed. And the list could go on!

Example 1.3.1 A Relation as a Subset

Let $A = \{1,2\}$ and $B = \{1,2,3\}$ and define a relation R from A to B as follows: Given any (x, y)∈ $A \times B$.

(x , y)∈ R means that $\frac{x-y}{2}$ is an integer.

a. State explicitly which ordered pairs are in $A \times B$ and which are in R.

Example 1.3.1 A Relation as a Subset

b. Is $1\ R\ 3$? Is $2\ R\ 3$? Is $2\ R\ 2$?

c. What are the domain and co-domain of R?

Example 1.3.2 The Circle Relation

Define a relation C from $\boldsymbol{R}$ to $\boldsymbol{R}$ as follows: For any (x, y)∈ $\boldsymbol{R} \times \boldsymbol{R}$,

(x, y)∈ means that $x^2 + y^2 = 1$

a. Is (1, 0) ∈ C? Is (0, 0)∈ C? Is $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$∈ C?

Is -2 C 0? Is 0 C (-1)? Is 1 C 1?

Example 1.3.2 The Circle Relation

b. What are the domain and co-domain of C

c. Draw a graph for C by plotting the points of C in the Cartesian plane.

Arrow Diagrams of Relations

Suppose R is a relation from a set A to a set B. The **arrow diagram for** R is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:

Draw an arrow from x to y

if, and only if, $x \ R \ y$

if, and only if, $(x, y) \in R$.

Example 1.3.3

Arrow Diagrams of Relations

Let $A = \{1,2,3\}$ and $B = \{1,3,5\}$ and define relation S and T from A to B as follows:

For all (x, y)∈ $A \times B$,

(x, y) ∈ S means that x < y s is "less than" relation.

$T = \{(2,1),(2,5)\}$

Draw arrow diagrams for S and T .

Functions

• Definition

A **function** F **from a set** A **to a set** B is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

• Notation

If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $\boldsymbol{F(x)}$, which is read "***F* of *x***."

Example 1.3.4 Functions and Relations on Finite Sets

Let $A = \{2,4,6\}$ and $B = \{1,3,5\}$. Which of the relations R, S, and T defined below are functions from A to B?

a. $R = \{(2,5),(4,1),(4,3),(6,5)\}$

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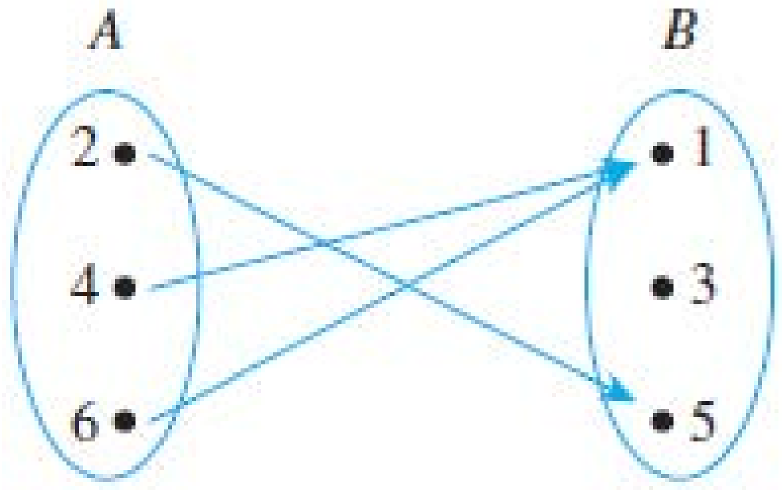
Example 1.3.4 Functions and Relations on Finite Sets

b. For all (x, y)∈ $A \times B$, (x, y)∈ S means that y = x+1.

Example 1.3.4 Functions and Relations on Finite Sets

c. T is defined by the arrow diagram.



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Example 1.3.5 Functions and Relations on Sets of Real Numbers

a. In example 1.3.2 the circle relation C was defined as follows:

For all (x, y)∈ $\boldsymbol{R} \times \boldsymbol{R}$, (x, y)∈ C means that $x^2 + y^2 = 1$

Is C a function? If it is, find $C(0)$ and $C(1)$.

Example 1.3.5 Functions and Relations on Sets of Real Numbers

b. Define a relation from $\boldsymbol{R}$ to $\boldsymbol{R}$ as follows:

For all (x, y)∈ $\boldsymbol{R} \times \boldsymbol{R}$, (x, y)∈ L means that y = x-1

Is L a function? If it is, find $L(0)$ and $L(1)$.

Function Machines

Another useful way to think of a function is as a machine. Suppose f is a function from X to Y and an input x of X is given. Imagine f to be a machine that processes x in a certain way to produce the output $f(x)$.

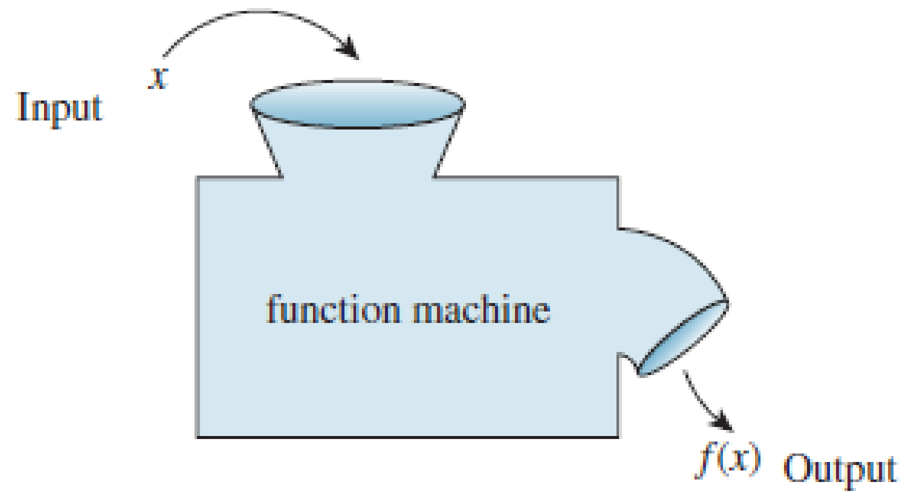

Figure 1.3.1

Example 1.3.7 Equality of Functions

Define $f: \boldsymbol{R} \to \boldsymbol{R}$ and $g: \boldsymbol{R} \to \boldsymbol{R}$ by the following formulas:

$$f(x) = |x| \qquad \text{for all } x \in \boldsymbol{R}$$

$$g(x) = \sqrt{x^2} \qquad \text{for all } x \in \boldsymbol{R}$$

Does $f = g$?