

Partial Fractions Decomposition

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are both polynomials. A rational expression is proper if the degree of the numerator is lower than the degree of the denominator; otherwise it is improper. An improper rational expression is the sum of a polynomial and a proper rational expression. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

For example

$$\frac{x^3 + x}{x + 2} = x^2 - 2x + 5 - \frac{10}{x + 2}$$

where $x^2 - 2x + 5$ is the quotient and -10 is the remainder upon division of the denominator into the numerator.

Partial fraction decomposition is the process of rewriting a rational expression as the sum of a quotient polynomial plus partial fractions. If the rational expression is proper, the quotient will be zero.

Example 1: Find the partial fraction decomposition of $\frac{x^3 + 2x^2 - 1}{x - 2}$.

After finding the quotient, the next step in decomposing a rational expression is factoring the denominator. It can be shown that any polynomial Q (the denominator) can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$). For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = x^4 - 16 = (x^2 + 4)(x^2 - 4) = (x^2 + 4)(x + 2)(x - 2)$$

The third step is to express the proper rational function $R(x)/Q(x)$ as a sum of **partial fractions** of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + c)^j}$$

CASE I: The denominator $Q(x)$ is a product of distinct linear factors.

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \dots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k} \quad (1)$$

Example 2: Find the partial fraction decomposition of

(a) $\frac{x - 9}{x^2 + 3x - 10}$

(b) $\frac{3x^2 - 7x - 2}{x^3 - x}$

CASE II: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation (1), we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r} \quad (2)$$

By way of illustration, we could write

$$\frac{x^3 + 4}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

Example 3: Find the partial fraction decomposition of

(a) $\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}$

(b) $\frac{x-2}{x^2(x-1)^2}$

CASE III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equation (1) and (2), the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c} \quad (3)$$

where A and B are constants to be determined. For instance, the function given by $x/[(x-3)(x^2+1)(x^2+2)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-3)(x^2+1)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+2}$$

Example 4: Find the partial fraction decomposition of

(a) $\frac{3x^2 - 4x + 3}{x^3 + x}$

(b) $\frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)}$

CASE IV: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction (3), the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r} \quad (4)$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the term in (4) can be integrated by first completing the square.

Example 5: Find the partial fraction decomposition of $\frac{6x^2 - 15x + 22}{(x+3)(x^2+2)^2}$.