EE320 Chapter 9
Optimization under Equality Constraint

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1 Effect of constraint

Suppose that we have a function with one choice variable and we want to find its maximum/minimum point:

unconstrained.opt

$$\boxed{\max_{x} y = -x^2 + 10x + 4}$$

FOC

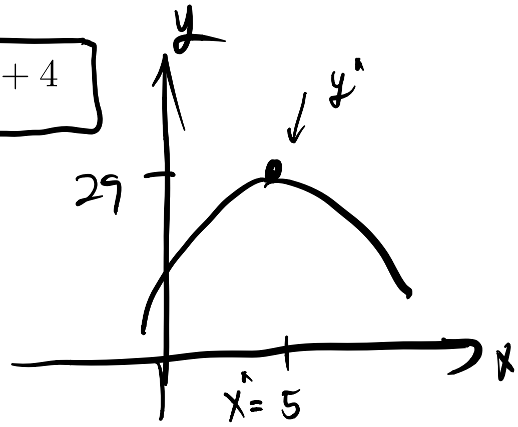
$$\boxed{\frac{dy}{dx}} = -2x + 10 = 0$$

$$x^* = 5$$

$$y^* = 29$$

SOC

$$\frac{d^2y}{dx^2} = -2 < 0 \quad \text{maximum}$$



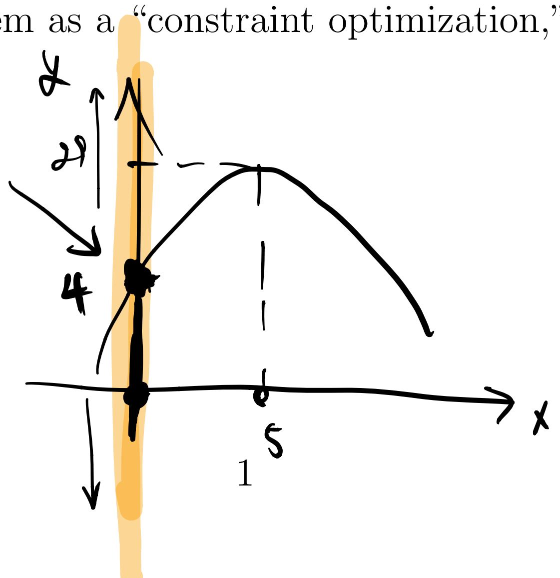
This method is called "unconstraint optimization." Consider a different case which we impose a certain condition on a function:

$$\boxed{\max_{x} y = -x^2 + 10x + 4 \text{ subject to } x = 0}$$

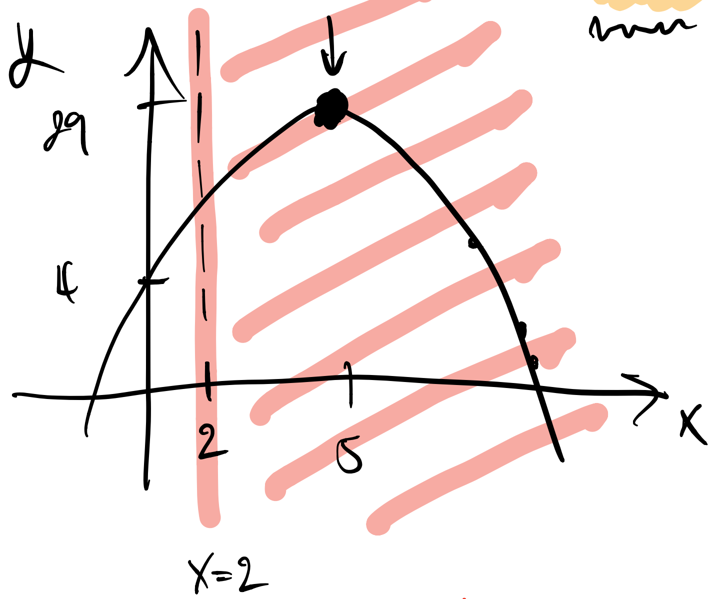
We call this problem as a "constraint optimization," when values of x are restricted.

x = 0⇒ y = 4

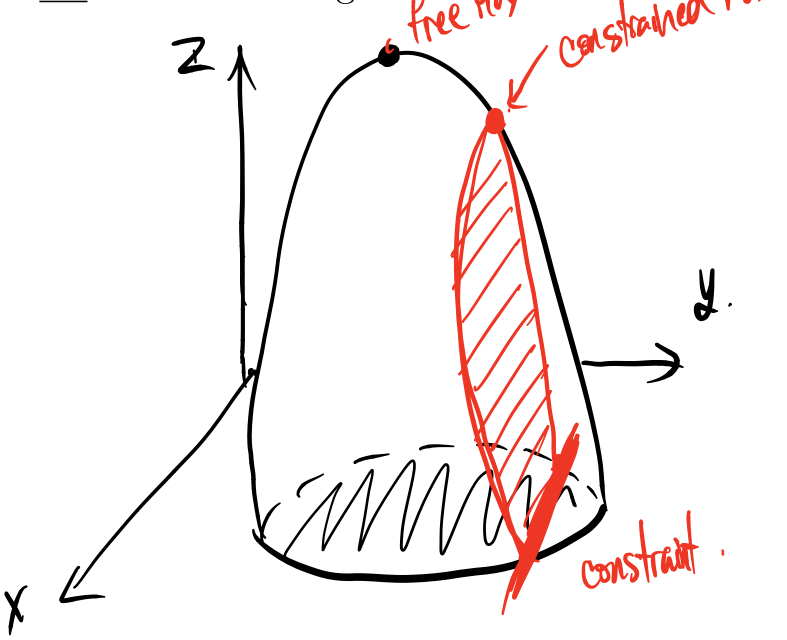
constrained opt.



In this case, we set $x = 0$, so we call it "equality constraint." If instead, we have the problem subject to an inequality $x \geq 2$, "inequality constraint."



ex. 3-dimension figure



Note maximum number of constraint = number of independent variable

ex.

1. $y = -x^2 + 10x + 4$ $\Rightarrow$ 1 independent variable = 1 maximum constraint (x)
2. $y = f(x_1, x_2)$ $\Rightarrow$ 2 ind variable = 2 maximum constraints (x_1, x_2)
3. $y = f(x_1, x_2, \dots, x_n)$ $\Rightarrow$ n ind variable = n maximum constraints ($x_1, x_2, \dots, x_n$)

$\max_{x_1, x_2} U = x_1 x_2 + 2x_1$

st $4x_1 + 2x_2 = 60$

2 Constraint Optimization

Suppose that a consumer has the following utility function: $U = x_1 x_2 + 2x_1$, given $p_1 = 4$ and $p_2 = 2$, this consumer's budget constraint is then: $4x_1 + 2x_2 = 60$. What will the optimal combination of goods x_1 and x_2 that give the highest utility subject to budget constraint be?

We can solve this problem using 2 approaches:

i) Substitution method: Substitute the constraint into objective function and solve for an optimal choices:

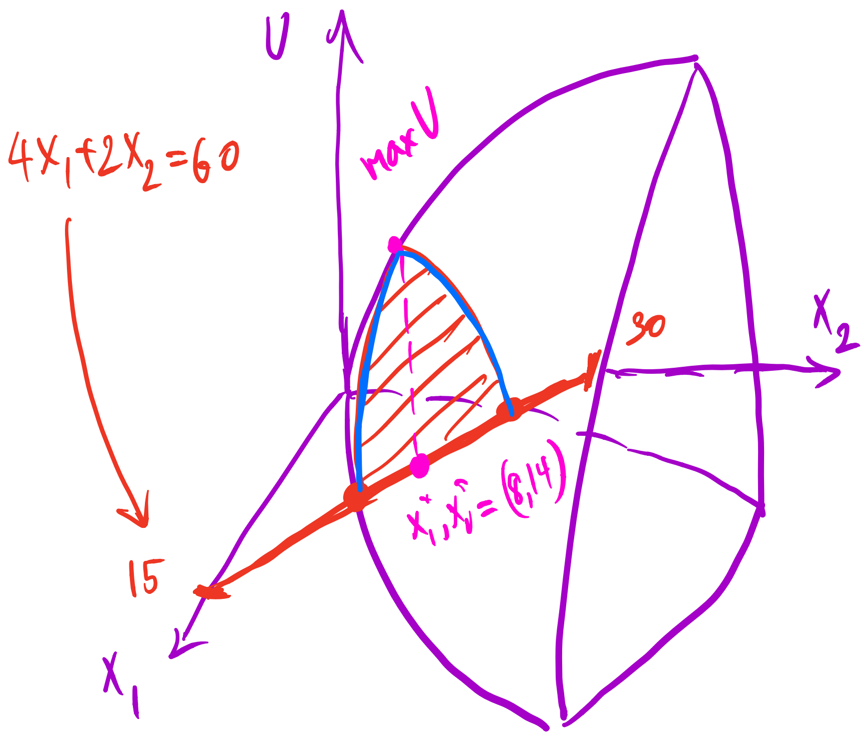
$$\boxed{4x_1 + 2x_2 = 60} \Rightarrow x_2 = 30 - 2x_1 \text{ substitute into utility function}$$

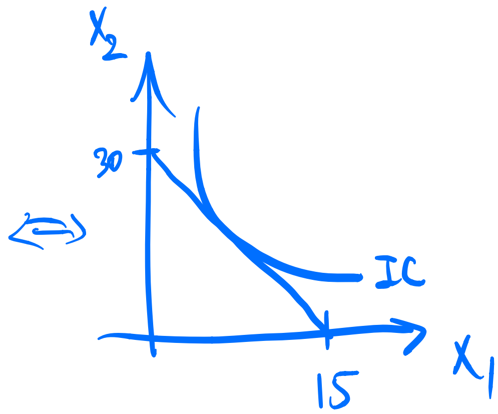
$$\Rightarrow U = x_1 x_2 + 2x_1 = x_1(30 - 2x_1) + 2x_1$$

$$= 32x_1 - 2x_1^2 \qquad x_2^* = 30 - 2x_1 = 14$$

FONC $\quad \frac{dU}{dx_1} = 32 - 4x_1 = 0 \Rightarrow x_1^* = 8$

SOSC $\quad \frac{d^2U}{dx_1^2} = -4 < 0$ ✓ Max.





max $U = x_1 x_2 + 2x_1$

st $4x_1 + 2x_2 = 60$

ii) Lagrange multiplier method: Convert a constraint optimization problem into an unconstraint optimization problem)

$$\boxed{\mathcal{L} = x_1 x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)}$$

$$\mathcal{L} = x_1 x_2 + 2x_1 - \lambda(4x_1 + 2x_2 - 60) \Rightarrow \quad \text{give the same answer}$$

$$\mathcal{L} = L(\lambda, x_1, x_2)$$

FOC : $\frac{\partial \mathcal{L}}{\partial \lambda} = 60 - 4x_1 - 2x_2 = 0$ --- ① BC

$\frac{\partial \mathcal{L}}{\partial x_1} = x_2 + 2 - 4\lambda = 0$ --- ② $\Rightarrow MU_1 = 4\lambda$

$\frac{\partial \mathcal{L}}{\partial x_2} = x_1 - 2\lambda = 0$ --- ③ $\Rightarrow MU_2 = 2\lambda$.

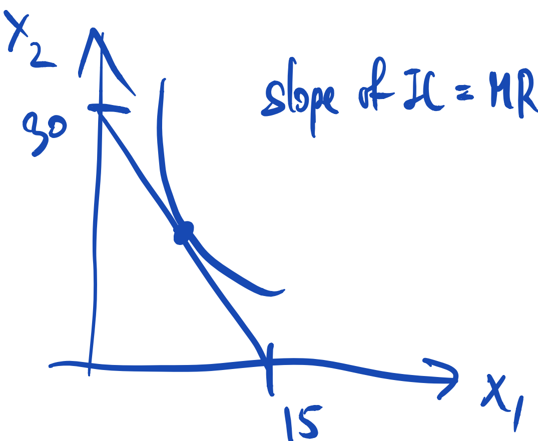
3 unknowns and 3 equations solve for λ^*, x_1^*, x_2^*

$$\left.\begin{aligned} x_2 + 2 &= 4\lambda \\ x_1 &= 2\lambda \end{aligned}\right\} \frac{②}{③} : \frac{x_2 + 2}{x_1} = 2 \Rightarrow \begin{aligned} x_2 + 2 &= 2x_1 \\ 60 - 4x_1 - 2x_2 &= 0 \end{aligned}$$

$\Downarrow$

$x_1^* = 8, \; x_2^* = 14$

$\boxed{\lambda^* = 4}$



slope of IC $= MRS = -\frac{MU_1}{MU_2} = \frac{-4\lambda}{2\lambda} = -2$

$4x_1 + 2x_2 = 60 \Rightarrow x_2 = 30 - 2x_1 \Rightarrow$ slope $= -2$.

FOC.

$\mathcal{L}_x = U_x - \lambda BC_x = 0. \Rightarrow \lambda = \frac{U_x}{BC_x}$

$\mathcal{L}_y = U_y - \lambda BC_y = 0 \Rightarrow \lambda = \frac{U_y}{BC_y}$

$\mathcal{L}_\lambda = BC(x,y) - I_0 = 0 \Rightarrow BC(x,y) = I_0$

1st

$\mathcal{L} = U(x,y) - \lambda(BC(x,y) - I_0)$

$\boxed{\frac{d\mathcal{L}}{dI} = \frac{dU}{dI} + \lambda}$

λ

FOC

$$\mathcal{L} = U(x,y) - \lambda\left(BC(x,y) - I_0\right)$$

$$d\mathcal{L} = U_x\,dx + U_y\,dy - \lambda BC_x\,dx - \lambda BC_y\,dy + \lambda\,dI_0$$

$$\frac{d\mathcal{L}}{dI_0} = U_x\frac{dx}{dI_0} + U_y\frac{dy}{dI_0} - \lambda BC_x\frac{dx}{dI_0} - \lambda BC_y\frac{dy}{dI_0} + \lambda.$$

$$\frac{d\mathcal{L}}{dI_0} = \left(U_x - \lambda BC_x\right)\frac{dx}{dI_0} + \left(U_y - \lambda BC_y\right)\frac{dy}{dI_0} + \lambda.$$

$$U_x = \lambda BC_x \qquad U_y = \lambda BC_y. \Rightarrow \boxed{\frac{d\mathcal{L}}{dI_0} = \lambda.} \Rightarrow \frac{dU}{dI} = \lambda.$$

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$\mathcal{L} = U(x,y) + \lambda(I_0 - BC(x,y))$

$\boxed{\frac{d\mathcal{L}}{dI} = \frac{dU}{dI} + \lambda}$

$\max_{x,y} V(x,y)$

st. $I_0 = BC(x,y) = P_x x + P_y y$

$\Rightarrow \mathcal{L} = V(x,y) + \lambda(I_0 - BC(x,y))$

3 Interpretation of Lagrange multiplier

Consider the problem $\max_{x,y} f(x,y)$ subject to $g(x,y) = c$

Solutions are $x^*(c), y^*(c), f^*(c)$ $\quad \mathcal{L} = f(x,y) + \lambda[c - g(x,y)]$

$f^*(c)$ is optimal value function AND $\boxed{\frac{d}{dc} f^*(c) = \lambda(c)}$

Lagrange multiplier λ is the rate at which the optimal value of the objective function changes $\Delta f(x,y)$ with respect to changes in the constraint constant Δc.

In economic application, c denotes the available stock of some resources, while $f(x,y)$ denotes utility, profit, etc.

$\underset{\text{utility}}{df^*(c)} = \lambda(c)\underset{\text{income}}{dc}$ is the change in utility or profit that can be obtained from dc units more of the resources.

Thus, λ is called a "shadow price" of the resource.

$\min_{K,L} C(K,L)$

st. $Q_0 = f(K,L)$

$\mathcal{L} = C(K,L) + \lambda(Q_0 - f(K,L))$

ex. for cost minimization problem, $\lambda = \frac{dC}{dQ_0} = MC$

utility max. $\lambda = \frac{dU}{dY_0} =$ MU of income

profit max. $\lambda = \frac{d\Pi}{dQ_0}$

output max. $\lambda = \frac{dQ}{dC_0}$

ex. Objective function: $z = xy$ subject to $x + y = 6$

$\mathcal{L} = xy + \lambda(6 - x - y)$

$\Rightarrow \boxed{x^* = 3, y^* = 3, \lambda^* = 3, z^* = 9}$

FOC

$$\mathcal{L}_x = y - \lambda = 0 \Leftrightarrow y = \lambda$$

$$\mathcal{L}_y = x - \lambda = 0 \Leftrightarrow x = \lambda$$

$$\mathcal{L}_\lambda = 6 - x - y = 0. \Leftrightarrow x + y = 6. \Rightarrow 2\lambda = 6$$

$$\left.\begin{array}{l} \lambda^* = 3 \\ x^* = 3 \\ y^* = 3 \end{array}\right\} z^* = 9.$$

ex. Objective function: $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $g(x_1, x_2) = x_1 + 4x_2 = 2$

$\mathcal{L} = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2)$

$\Rightarrow x_1^* = \frac{2}{17}, x_2^* = \frac{8}{17}$

4 Second-order conditions for constrained optimization

$$\begin{aligned}
\text{Recall the FONC for } L(x,y) &= f(x,y) + \lambda[c - g(x,y)] \\
L_x(x,y) &= f_x(x,y) - \lambda g_x(x,y) \\
L_y(x,y) &= f_y(x,y) - \lambda g_y(x,y) \\
\lambda &= \frac{f_x}{g_x} = \frac{f_y}{g_y}
\end{aligned}$$

Find SOSC

$$\begin{aligned}
L_{xx}(x,y) &= f_{xx} - \lambda g_{xx} = f_{xx} - \frac{f_y}{g_y} g_{xx} \\
L_{yy}(x,y) &= f_{yy} - \lambda g_{yy} = f_{yy} - \frac{f_y}{g_y} g_{yy} \\
L_{xy}(x,y) &= f_{xy} - \lambda g_{xy} = f_{xy} - \frac{f_y}{g_y} g_{xy}
\end{aligned}$$

Thus, d^2z can be written as

$$\begin{aligned}
d^2z &= L_{xx}dx^2 + 2L_{xy}dxdy + L_{yy}dy^2 \\
&= \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} L_{xx} & L_{xy} \\ L_{yx} & L_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}
\end{aligned}$$

SOSC conditions are:

Max of z : d^2z is negative definite ($d^2z<0$) , st. dg = 0

Min of z : d^2z is positive definite ($d^2z>0$) , st. dg = 0

where $dg = g_x dx + g_y dy = 0$

Define the determinant of the Bordered Hessian matrix as

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

$$\mathcal{L} = \underbrace{f(x,y)}_{\text{obj}} + \lambda\left(c - \underbrace{g(x,y)}_{\text{constraint}}\right)$$

$$g_1 = \frac{\partial g}{\partial x} \qquad \mathcal{L}_{11} = \frac{\partial^2 \mathcal{L}}{\partial x^2} \qquad \mathcal{L}_{12} = \frac{\partial^2 \mathcal{L}}{\partial x \partial y}.$$

$$g_2 = \frac{\partial g}{\partial y} \qquad \mathcal{L}_{22} = \frac{\partial^2 \mathcal{L}}{\partial y^2}$$

or SOSC conditions are

Max: d^2z is negative definite st. $g_x dx + g_y dy = 0$ iff $|\bar{H}| > 0$

Min: d^2z is positive definite st. $g_x dx + g_y dy = 0$ iff $|\bar{H}| < 0$

$g(x_1, x_2) = 4x_1 + 2x_2 = 60$

ex. Opt. $U = x_1x_2 + 2x_1$ subject to $4x_1 + 2x_2 = 60$. Check whether it is max or min value.

$g(x_1, x_2) = c$

$\frac{\partial g}{\partial x_1} = \frac{\partial}{\partial x_1}(4x_1 + 2x_2) = 4$

$\frac{\partial g}{\partial x_2} = \frac{\partial}{\partial x_2}(4x_1 + 2x_2) = 2$

$$\mathcal{L} = x_1x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)$$

FONC

$$\left.\begin{aligned} \mathcal{L}_1 &= x_2 + 2 - 4\lambda = 0 \\ \mathcal{L}_2 &= x_1 - 2\lambda = 0 \\ \mathcal{L}_\lambda &= 60 - 4x_1 - 2x_2 = 0 \end{aligned}\right\} \quad x_1^* = 8,\ x_2^* = 14$$

SOSC

$0 + 0 + 0$

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 4 & 2 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$0 + 8 + 8$

$\therefore |\bar{H}| = 16 > 0$ Max.

$U^*(8, 14) = 128$ is a maximum utility subject to budget constraint.

ex. $f(x_1, x_2) = x_1^* + x_2^*$ subject to $g(x_1, x_2) = x_1 + 4x_2 = 2$ Find their critical points and check it is max or min.

SOSC $\quad |\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -84 < 0.$

$\Rightarrow$ minimum.

5 Economic Applications of Constrained Optimization

Application: Utility maximization Problem

Objective function: $\max\limits_{x_1, x_2} U = U(x_1, x_2)$

Subject to $Y_0 = p_1 x_1 + p_2 x_2$

$\mathcal{L} = U(x_1, x_2) + \lambda(Y_0 - p_1 x_1 - p_2 x_2)$

FONC

$$\begin{aligned} L_1 &= u_1(x_1, x_2) - \lambda p_1 = 0 && \Leftrightarrow u_1 = \lambda p_1 \\ L_2 &= u_2(x_1, x_2) - \lambda p_2 = 0 && \Leftrightarrow u_2 = \lambda p_2 \\ L_\lambda &= Y_0 - p_1 x_1 - p_2 x_2 = 0 \end{aligned}$$

$\Rightarrow \frac{u_1}{u_2} = \frac{p_1}{p_2}$

$D(x_1^*, x_2^*)$ is such that $\frac{u_1(x_1,x_2)}{u_2(x_1,x_2)} = \frac{P_1}{P_2}$ and $Y_0 = p_1 x_1 + p_2 x_2$

from $\left.\begin{aligned} \lambda &= \frac{u_1}{p_1} \\ \lambda &= \frac{u_2}{p_2} \end{aligned}\right\} \boxed{\frac{u_1}{p_1} = \frac{u_2}{p_2}} \Rightarrow \frac{u_1}{u_2} = \frac{p_1}{p_2}$

diminishing marginal utility.

$\frac{u_1}{p_1} = \frac{u_2}{p_2}$: $\quad \frac{u_1}{p_1} > \frac{u_2}{p_2} \Rightarrow x_1 \uparrow \Rightarrow u_1 \downarrow \Rightarrow \frac{u_1}{p_1} = \frac{u_2}{p_2}$

$\frac{u_1}{p_1} < \frac{u_2}{p_2} \Rightarrow x_2 \uparrow \Rightarrow u_2 \downarrow \Rightarrow \frac{u_1}{p_1} = \frac{u_2}{p_2}$ //

$$g(x_1, x_2) = p_1x_1 + p_2x_2$$

$$\begin{cases} g_1 = p_1 \\ g_2 = p_2 \end{cases}$$

SOSC

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & u_{11} & u_{21} \\ p_2 & u_{21} & u_{22} \end{vmatrix} = -p_1^2u_{11}^2 + 2p_1p_2u_{12} - p_2^2u_{11}^2$$

ex. $U = 50A + 200B - 0.5A^2 - 2.5B^2$. Find A,B that maximize the utility given

$$\left.\begin{array}{rcl} P_A & = & 10 \\ P_B & = & 5 \\ \bar{Y} & = & 490 \end{array}\right\} 490 = 10A + 5B$$

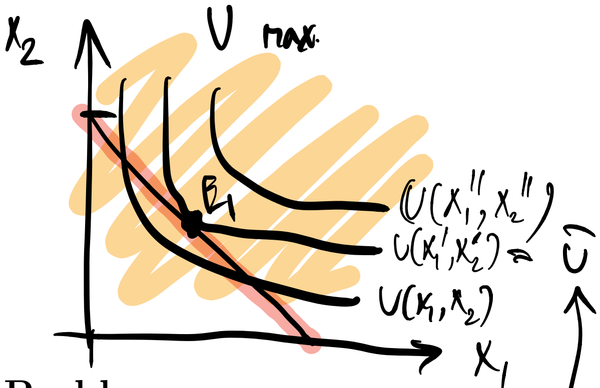
$\therefore$ max $U = 50A + 200B - 0.5A^2 - 2.5B^2$

subject to 10A + 5B = 490

$$\mathcal{L} = 50A + 200B - 0.5A^2 - 2.5B^2 + \lambda(490 - 10A - 5B)$$

$$\begin{array}{rcl} A^* & = & 30 \\ B^* & = & 38 \\ U^*(A^*, B^*) & = & 5,040 \end{array}$$

$|\bar{H}| = 525 > 0$ max.



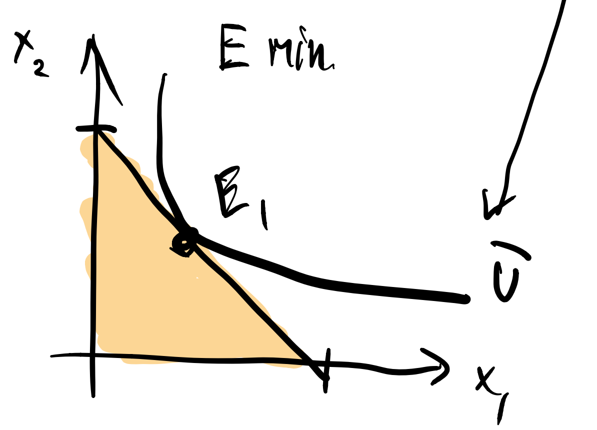
Duality

Application: Expenditure Minimization Problem

Objective function $\min_{x_1,x_2} \boxed{E = p_1x_1 + p_2x_2}$

Subject to $\bar{U} = U(x_1, x_2)$

$$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(\bar{U} - U(x_1, x_2))$$



FONC

$$\left.\begin{array}{rcccl} L_1 & = & p_1 - \lambda u_1(x_1, x_2) & = & 0 \\ L_2 & = & p_2 - \lambda u_2(x_1, x_2) & = & 0 \\ L_\lambda & = & \bar{u} - u(x_1, x_2) & = & 0 \end{array}\right\} \boxed{\frac{u_1}{u_2} = \frac{p_1}{p_2}} \text{ and } \bar{u} = u(x_1, x_2)$$

<u>SOSC</u> $|\bar{H}| < 0$ min.

<u>ex.</u> $U = 50A + 200B - 0.5A^2 - 2.5B^2$ and consumer would like to have a utility fixed at $\bar{u} = 5{,}040$. Find A and B that minimize the expenditure given that $P_A = 10$ and $P_B = 5$

$\therefore$ min $E = 10A + 5B$ subject to $5,040 = 50A + 200B - 0.5A^2 - 2.5B^2$

$$\begin{aligned} \Rightarrow A^* &= 30 \\ B^* &= 38 \\ E^* &= 490 \end{aligned}$$

Duality: Given that household wants to maximize his/her utility function given the budget constraint, that particular household can also minimize the expenditure given the desired level of utility. We claim that the solutions for expenditure minimization and utility maximization are the same. Let's prove this statement:

<u>ex.</u> $u = x_1 x_2$ given p_1, p_2, y_0

$\max_{x_1,x_2} u = x_1 x_2$ subject to $p_1 x_1 + p_2 x_2 = y_0$

$\mathcal{L} = x_1 x_2 + \lambda(y_0 - p_1 x_1 - p_2 x_2)$

FOC:

$$\begin{aligned} \mathcal{L}_1 &= x_2 - \lambda p_1 = 0 && \Rightarrow x_2 = \lambda p_1 \\ \mathcal{L}_2 &= x_1 - \lambda p_2 = 0 && \Rightarrow x_1 = \lambda p_2 \\ \mathcal{L}_\lambda &= p_1 x_1 + p_2 x_2 = y_0 \end{aligned}$$

$\therefore \frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2} x_1$

$\Rightarrow p_1 x_1^* + p_2 (\frac{p_1}{p_2} x_1^*) = y_0$

$$2p_1 x_1^* = y_0$$

utility, profit, ...

$\max_{x,y}\ f(x,y)$

st. $C_0 = B(x,y)$

① Substitution method   BC, production fn, ...

$C_0 = B(x,y) \rightarrow f(x,y)$

$\left.\begin{array}{l}\text{FONC} \\ \text{SOSC} \rightarrow \text{Hessian}\end{array}\right\}$ ch 8.

② $\mathcal{L} = f(x,y) + \lambda[C_0 - B(x,y)]$

FONC $\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial y}, \frac{\partial \mathcal{L}}{\partial \lambda}$.

SOSC $|\bar{H}| > 0$ max $|\bar{H}| = \begin{vmatrix} 0 & B_1 & B_2 \\ B_1 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ B_2 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix}$

$|\bar{H}| < 0$ min.

$$\Rightarrow x_1^* = \frac{I}{2p_1}$$

$$x_2^* = \frac{I}{2p_2}$$

U max.

$$\min_{x_1,x_2} E = p_1x_1 + p_2x_2$$

st. $x_1 + x_2 = \bar{u}$

$$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(\bar{u} - x_1x_2)$$

FOC:

$$\begin{aligned}
\mathcal{L}_1 &= p_1 - \lambda x_2 = 0 \ \Rightarrow \ p_1 = \lambda x_2 \\
\mathcal{L}_2 &= p_2 - \lambda x_1 = 0 \ \Rightarrow \ p_2 = \lambda x_1 \\
\mathcal{L}_\lambda &= x_1x_2 = \bar{u}
\end{aligned}$$

$$\therefore \frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2}x_1$$

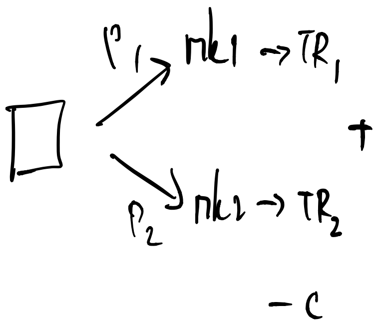
$$\bar{u} = x_1^* \cdot \left(\frac{p_1}{p_2}x_1^*\right)$$

$$\Rightarrow x_1^* = \sqrt{\left(\frac{p_2}{p_1}\bar{u}\right)}$$

$$x_2^* = \sqrt{\left(\frac{p_1}{p_2}\bar{u}\right)}$$

E min.

Application: Profit Maximization Problem



Objective function $\max_{q_1,q_2} \Pi = \underline{p_1}q_1 + \underline{p_2}q_2 - c(q_1, q_2)$

Subject to $\boxed{\bar{Q}_0 = q_1 + q_2}$ ←

$$\mathcal{L} = p_1q_1 + p_2q_2 - c(q_1, q_2) + \lambda(\bar{Q} - q_1 - q_2)$$

<u>FONC</u>

$$\begin{array}{ll}\{q_1\} & L_1 = p_1 - c_1(q_1, q_2) - \lambda = 0 \\ \{q_2\} & L_2 = p_2 - c_2(q_1, q_2) - \lambda = 0\end{array}\Bigg\} \boxed{p_1 - c_1 = p_2 - c_2 \text{ and } \bar{Q} = q_1 + q_2}$$

$$\{\lambda\} \quad L_\lambda = \bar{Q} - q_1 - q_2 = 0$$

$\frac{\partial\, q_1+q_2}{\partial q_1}$ $\frac{\partial\, q_1+q_2}{\partial q_2}$

SOSC $|\bar{H}|>0$ max.

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & \alpha_{11} & \alpha_{12} \\ 1 & \alpha_{21} & \alpha_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & c_{11} & c_{12} \\ 1 & c_{12} & c_{22} \end{vmatrix}$$ if >0, max

ex. Suppose that $p_1 = 80$, $p_2 = 100$, $TC = 100 + 0.1q_1^2 + 0.2q_2^2$ and $q_1 + q_2 = 325$. Find q_1 and q_2 that maximize profit.

$$\boxed{\max_{q_1,q_2} = 80q_1 + 100q_2 - 100 - 0.1q_1^2 - 0.2q_2^2 \text{ subject to } q_1 + q_2 = 325}$$

$$\Rightarrow \mathcal{L} = \underbrace{80q_1 + 100q_2 - 100 - 0.1q_1^2 - 0.2q_2^2}_{\pi} + \underbrace{\lambda(325 - q_1 - q_2)}_{\text{constraint.}}$$

FONC

$$L_1 = \frac{\partial L}{\partial q_1} = 80 - 0.2q_1 - \lambda = 0$$
$$L_2 = \frac{\partial L}{\partial q_2} = 100 - 0.4q_2 - \lambda = 0$$

$80 - 0.2q_1 = \lambda$

$100 - 0.4q_2 = \lambda$

$-0.2q_1 + 80 = -0.4q_2 + 100$

$\Downarrow$

$-0.2q_1 + 0.4q_2 - 20 = 0.$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = 325 - q_1 - q_2 = 0$$

$$\boxed{\begin{aligned} -q_1 + 2q_2 - 100 &= 0. \\ -q_1 - q_2 + 325 &= 0 \end{aligned}}$$

$3q_2 - 425 = 0.$

$q_2^* = \frac{425}{3}$

$q_1^* = 325 - \frac{425}{3}$

SOSC $\quad |\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -4 \end{vmatrix} = 6 > 0$

Max.

$\{q_1, q_2\}$

max π st. $\bar{Q}$ $\neq$ max Q st. Cost.

Application: Output Maximization Problem $\{K, L\}$

Objective function $\max_{K,L} Q = Q(K, L)$

Subject to $\bar{C} = wL + rK$

$\mathcal{L} = Q(K, L) + \lambda(\bar{C} - wL - rK)$

FONC

$$\begin{aligned} L_K &= Q_K - \lambda r = 0 \\ L_L &= Q_L - \lambda w = 0 \end{aligned} \Bigg\} \boxed{\frac{Q_K}{Q_L} = \frac{r}{w}}$$

$$L_\lambda = C_0 - wL - rK = 0$$

We can find $Q^*(K^*, L^*)$, K^* , L^*

SOSC Also check $|\bar{H}| = \begin{vmatrix} 0 & r & w \\ r & Q_{KK} & Q_{LK} \\ w & Q_{KL} & Q_{LL} \end{vmatrix}$

ex. Suppose $\boxed{Q = KL}$, w = 6 , r = 10 , $\bar{C}$ = 60. Find K* and L* that maximize output st $\bar{C} = 60$

$\mathcal{L}_K = L - 10\lambda$

$\mathcal{L}_L = K - 6\lambda$

$\mathcal{L}_\lambda = 60 - 6L - 10K$

$\boxed{\max_{K,L} Q = KL}$ subject to $60 = 6L + 10K$

$\mathcal{L}$ = KL + λ (60-6L-10K) $\Rightarrow$ $L^* = 5$ $K^* = 3$

$|\bar{H}| = 120 > 0$ max.

$\{K^* = 3, L^* = 5, Q^* = 15\}$

$\bar{C} = 60$

max $Q(K,L)$
st. $\bar{C}$

Duality.

min $C(K,L)$
st. $\bar{Q}$

$$|\bar{H}| = \begin{vmatrix} 0 & 10 & 6 \\ 10 & \mathcal{L}_{11} & \mathcal{L}_{12} \\ 6 & \mathcal{L}_{21} & \mathcal{L}_{22} \end{vmatrix} = \begin{vmatrix} 0 & 10 & 6 \\ 10 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix}$$

Application: Cost Minimization Problem

Objective function $\boxed{\min_{K,L} C = wL + rK}$

Subject to $\boxed{\bar{Q} = f(K,L)}$

$\mathcal{L} = wL + rK + \lambda(\bar{Q} - f(K,L))$

FONC

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial K} = L_K &= r - \lambda f_K = 0 \\ \frac{\partial \mathcal{L}}{\partial L} = L_L &= w - \lambda f_L = 0 \end{aligned}\right\} \frac{f_K}{f_L} = \frac{r}{w} \qquad \Rightarrow \qquad \boxed{C^*(K^*, L^*) \ , \ K^* \ , \ L^*}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = L_\lambda = \bar{Q} - f(K,L) = 0$$

minimum cost

$C(K^*, L^*)$

subject to $\bar{Q}$

$\bar{Q} = f(K,L)$ $\to$ $\frac{\partial f}{\partial K} = f_K$, $\frac{\partial f}{\partial L} = f_L$

<u>SOSC</u> $|\bar{H}| = \begin{vmatrix} 0 & f_K & f_L \\ f_K & L_{KK} & L_{LK} \\ f_L & L_{KL} & L_{LL} \end{vmatrix} < 0$ min.

$\bar{Q} = 15$

<u>ex.</u> Suppose Q = KL. A firm will produce 15 units of goods. Find K* and L* that minimize total cost given that w = 6 , r = 10.

Min C(w,r)=6L +10K (5; 30 + 30)

st. 15 = KL

$$\mathcal{L} = 6L + 10K + \lambda(15 - KL)$$

FOC: $\mathcal{L}_K, \mathcal{L}_L, \mathcal{L}_\lambda = 0 \Rightarrow K^*, L^*, \lambda^*$

SOC: $|\bar{H}|$

$$\Rightarrow \begin{aligned} K^* &= 3 \\ L^* &= 5 \\ |\bar{H}| &= -60 < 0 \end{aligned}$$

min.

$\{K^*=3, L^*=5, C^*=60\}$ $\bar{Q} = 15$

6 n-choice Variable and Multi-Constraint Cases

6.1 One Constraint

Objective function z = $f(\underbrace{x_1, x_2, ..., x_n}_{\#n})$

Subject to $\boxed{g(x_1, x_2, ..., x_n) = c}$ (one constraint)

$$\mathcal{L}(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) + \lambda\Big[c - g(x_1, x_2, ..., x_n)\Big]$$

<u>FONC</u>

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} = L_1 &= f_1 - \lambda g_1 = 0 \\ \frac{\partial L}{\partial x_2} = L_2 &= f_2 - \lambda g_2 = 0 \\ &\vdots \\ \frac{\partial L}{\partial x_n} = L_n &= f_n - \lambda g_n = 0 \\ \frac{\partial L}{\partial \lambda} = L_\lambda &= c - g(x_1, x_2, ..., x_n) = 0 \end{aligned}\right\} \text{(n+1) conditions}$$

n equations; +1

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

<u>SOSC</u> $\quad |\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}_{(n+1)\times(n+1)}$

d^2z is positive definite st. dg $= 0$ if and only if $|\bar{H}_2|, |\bar{H}_3|, ..., |\bar{H}_n| < 0$ always negative $\Rightarrow$ $\boxed{\text{Min}}$

d^2z is negative definite st. dg $= 0$ iff $|\bar{H}_2| > 0$

$|\bar{H}_3| < 0$

$$|\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}$$

$\vdots$

$(-1)^n|\bar{H}_n| > 0 \quad$ alternate signs $\Rightarrow$ $\boxed{\text{Max}}$

<u>ex.</u> min C = wL +iK + rT $\quad$ find K , L and T that minimize cost

st. $\bar{Q}$ = f(K,L,T)

$\mathcal{L} = wL + iK + rT + \lambda(\bar{Q} - f(K, L, T))$

6.2 k Equality Constraints

Objective function $z = f(x_1, x_2, ..., x_n)$ subject to constraints:

$$\begin{array}{lrcll}
1\text{st} & g^1(x_1, x_2, ..., x_n) & = & c_1 \leftarrow & \{\lambda_1\} \\
2\text{nd} & g^2(x_1, x_2, ..., x_n) & = & c_2 \leftarrow & \{\lambda_2\} \\
\vdots & \vdots & & \vdots & \vdots \\
k\text{th} & g^k(x_1, x_2, ..., x_n) & = & c_k \leftarrow & \{\lambda_k\}
\end{array}$$

$$\Rightarrow \mathcal{L} = f(x_1, x_2, ..., x_n) + \lambda_1[c_1 - g^1(\cdot)] + \lambda_2[c_2 - g^2(\cdot)] + \cdots + + \lambda_k[c_k - g^k(\cdot)]$$

$$\mathcal{L} = f(x_1, x_2, ..., x_n) + \sum_{j=1}^{k} \lambda_j[c_j - g^j(x_1, x_2, ..., x_n)]$$

<u>FONC</u>

$$\left.\begin{array}{l}
\frac{\partial L}{\partial x_1} = f_1 - \lambda_1 \frac{\partial g^1}{\partial x_1} - \lambda_2 \frac{\partial g^2}{\partial x_1} - \cdots - \lambda_k \frac{\partial g^k}{\partial x_1} = 0 \\
\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \qquad\quad \vdots \\
\frac{\partial L}{\partial x_n} = f_n - \lambda_1 \frac{\partial g^1}{\partial x_n} - \lambda_2 \frac{\partial g^2}{\partial x_n} - \cdots - \lambda_k \frac{\partial g^k}{\partial x_n} = 0
\end{array}\right\} \text{n equations}$$

$$\left.\begin{array}{l}
\frac{\partial L}{\partial \lambda_1} = c_1 - g^1(x_1, x_2, ..., x_n) = 0 \\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\frac{\partial L}{\partial \lambda_k} = c_k - g^k(x_1, x_2, ..., x_n) = 0
\end{array}\right\} \text{k equations}$$

<u>SOSC</u>

k columns   n columns

$$\begin{array}{rc|l}
 & \overbrace{\qquad\qquad\qquad}^{k\ \text{columns}} \quad \overbrace{\qquad\qquad\qquad}^{n\ \text{columns}} & \\
\text{k rows.} & \bar{H} = \left|\begin{array}{cccc|cccc}
0 & 0 & \cdots & 0 & g_1^1 & g_2^1 & \cdots & g_n^1 \\
0 & 0 & \cdots & 0 & g_1^2 & g_2^2 & \cdots & g_n^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & g_1^k & g_2^k & \cdots & g_n^k \\
\hline
g_1^1 & g_1^2 & \cdots & g_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\
g_2^1 & g_2^2 & \cdots & g_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
g_n^1 & g_n^2 & \cdots & g_n^k & L_{n1} & L_{n2} & \cdots & L_{nn}
\end{array}\right|_{(n+k)\times(n+k)} & \begin{array}{l}\text{first constraint} \\ \text{second} \\ \\ \text{kth constraint.}\end{array} \text{ k rows.} \\
\text{n rows.} & &
\end{array}$$

k×k

first c.   second c   kth c.

Consider (n-k) bordered leading principal minors:

$|\bar{H}_{k+1}|, |\bar{H}_{k+2}|, \dots, |\bar{H}_n|$

Max of z, sign $|\bar{H}_{k+1}| = (-1)^{k+1}$ alternate sign for $|\bar{H}_{k+2}|$ and so on.

Min of z, sign $\cancel{|\bar{H}_k|}$ $|\bar{H}_{k+1}|$ $= (-1)^k$ take the same sign

Step. 1 check n-k.
2. max. $(-1)^{k+1}$ alternate
min $(-1)^k$ same

ex. $z = f(x_1, x_2, x_3)$

st. $g(x_1, x_2, x_3) = c$ and $h(x_1, x_2, x_3) = d$

$\mathcal{L} = f(x_1, x_2, x_3) + \lambda_1[c - g] + \lambda_2[d - h]$

ex $n=5, k=1 \Rightarrow n-k=4$

FONC:

min $(k=1) \Rightarrow |\bar{H}_{k+1}| = |\bar{H}_2| : (-1)^1 < 0$

$|\bar{H}_2| < 0$
$|\bar{H}_3| < 0$
$|\bar{H}_4| < 0$
$|\bar{H}_5| < 0$.

max. $k=1 \Rightarrow |\bar{H}_{k+1}| = |\bar{H}_2| : (-1)^{k+1} = 1 > 0$.

$|\bar{H}_2| > 0$
$|\bar{H}_3| < 0$
$|\bar{H}_4| > 0$
$|\bar{H}_5| < 0$

SOSC: n = 3, k= 2 $\Rightarrow$ n-k = 1 only one bordered Hessian matrix

$|\bar{H}_n| = |\bar{H}_3|$ or $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_n|$

$|\bar{H}_{k+1}| = |\bar{H}_3| : (-1)^{2+1} = -1 < 0.$

$$\therefore |\bar{H}_3| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}_{5\times 5}$$

max if $|\bar{H}_{k+1}| = |\bar{H}_3| < 0$ because sign of $|\bar{H}_3| = (-1)^{2+1} = -1 < 0$ should be negative

min.

~~max~~ if $|\bar{H}_{k+1}| = |\bar{H}_3| > 0$ because sign of $|\bar{H}_3| = (-1)^2 = 1 > 0$ should be positive

$|\bar{H}| : (-1)^k = (-1)^2 = 1 > 0.$

# of variables   # of constraints

ex. $z = f(x_1, x_2, x_3, x_4)$ $n=4$, $k=2$

st. $g(x_1, x_2, x_3, x_4) = c$

$h(x_1, x_2, x_3, x_4) = d$

FONC: $\mathcal{L} = f(x_1, \dots, x_4) + \lambda_1 (c - g) + \lambda_2 (d - h)$

$$\left(\begin{array}{ll} \mathcal{L}_1 = 0 & \mathcal{L}_{\lambda_1} = 0 \\ \mathcal{L}_2 = 0 & \mathcal{L}_{\lambda_2} = 0 \\ \mathcal{L}_3 = 0 & \\ \mathcal{L}_4 = 0 & \text{6 equations} \end{array} \right) \Rightarrow x_1^*, x_2^*, x_3^*, x_4^*, \lambda_1^*, \lambda_2^*$$

6 unknowns.

SOSC   n = 4, k = 2 ⇒ n - k = 2 bordered Hessian matrix.

Step.
1. $n-k = 2$. ✓

Consider $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_3| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}$

2.1 Max. $(-1)^{k+1} = (-1)^3 < 0$

✓ $|\bar{H}_{k+1}| = |\bar{H}_3| < 0$

✓ $|\bar{H}_4| > 0$.

$|\bar{H}_{k+2}| = |\bar{H}_{2+2}| = |\bar{H}_4| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & h_1 & h_2 & h_3 & h_4 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ g_4 & h_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{vmatrix}_{6\times 6}$

2.2 min. $(-1)^k = (-1)^2 = 1 > 0$.

✓ $|\bar{H}_3| > 0$

✓ $|\bar{H}_4| > 0$.

max if sign of $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_3| = (-1)^{2+1} = -1$ negative
or $|\bar{H}_3| < 0$ and $|\bar{H}_{k+2}| = |\bar{H}_4| > 0$

min if sign of $|\bar{H}_3| = (-1)^2 = 1$ or $|\bar{H}_3| > 0$ and $|\bar{H}_{k+2}| = |\bar{H}_4| > 0$

$\bar{H} = \begin{vmatrix} 0 & \frac{\partial g}{\partial x_i} \\ \frac{\partial g}{\partial x_i}^T & L_{ij} \end{vmatrix}_{(n+k)\times(n+k)}$ for $j = 1, \ldots, n$.

ex. Given objective function $y = f(x_1, \ldots, x_{35})$ and 22 constraints

1. What is the dimension of $\bar{H}$? $|\bar{H}|_{(n+k)\times(n+k)} = |\bar{H}|_{(35+22)\times(35+22)} = |\bar{H}|_{57\times 57}$

   $n = 35$

   $k = 22$

2. What is the dimension of zero matrix inside $\bar{H}$?

   $= 0_{22\times 22}$ $\quad \bar{H} = \begin{vmatrix} 0_{22\times 22} & \frac{\partial g}{\partial x_i} \\ \frac{\partial g}{\partial x_i}^T & L_{ij} \end{vmatrix}$

3. Write SOCs for max.

$|\bar{H}_{k+1}| : (-1)^{k+1} = (-1)^{22+1} < 0$

step. 1. $(n-k) = 35 - 22 = 13$

2. max : $(-1)^{k+1} \Rightarrow |\bar{H}_{23}| < 0$, $|\bar{H}_{24}| > 0$, $\ldots$, $|\bar{H}_{35}|$

1st, 2nd, ..., 13th

$|\bar{H}_{n=35}|$

Min ...

7 Other Economic Application

max/min $f(x_1, \ldots, x_n)$

st. $g^1(\cdot) = c_1$

$\vdots$

$g^k(\cdot) = c_k$

FONC. x^*, λ^*

SOSC step.

1. $(n-k)$

2.1 max $(-1)^{k+1}$ alter

2.2 min $(-1)^k$ same

7.1 Homogeneous function

CRTS

$$F(\lambda x, \lambda y) = \lambda^r F(x, y) \rightarrow r = 1 : F(\lambda x, \lambda y) = \lambda F(x, y)$$

where r is a degree of homogeneous function.

r can indicate 3 types of return to scales (IRTS, CRTS, DRTS)

$r > 1 : F(\lambda x, \lambda y) = \lambda^r F(x, y)$ IRTS.

$r < 1 : F(\lambda x, \lambda y) = \lambda^r F(x, y)$ DRTS.

ex. $\boxed{Q = KL}$

$K_0, L_0 \Rightarrow Q_0 = K_0 L_0$

$\lambda K_0, \lambda L_0 \Rightarrow Q_1 = (\lambda K_0)(\lambda L_0) = \lambda^2 K_0 L_0 = \lambda^2 Q_0 \Rightarrow r = 2$:

If r = 2, then if we increase both L and K by 2 times, then the level of output would be rising by $\lambda^2 = 2^2 = 4$ times = IRTS

$Q = KL$: IRTS.

$Q = K + L$

$\{K_0, L_0\} \Rightarrow Q_0 = K_0 + L_0$

$\{\lambda K_0, \lambda L_0\} \Rightarrow Q = \lambda K_0 + \lambda L_0 = \lambda\underbrace{(K_0 + L_0)} = \lambda^1 Q_0$ CRTS. $r=1$

ex. $\boxed{Q = K+L}$

$Q(\lambda K, \lambda L) = \lambda K_0 + \lambda L_0 = \lambda(K+L) = \lambda Q_0(K,L) \Rightarrow$ CRTS

ex. $\boxed{Q = K^\alpha L^\beta}$ What should the value of α and β be such that Q has a property of CRTS?

CRTS $Q(\lambda K, \lambda L) = \lambda^1 Q(K,L)$ $r=1$

$\{K_0, L_0\} \Rightarrow Q(K_0, L_0) = K_0^\alpha L_0^\beta$

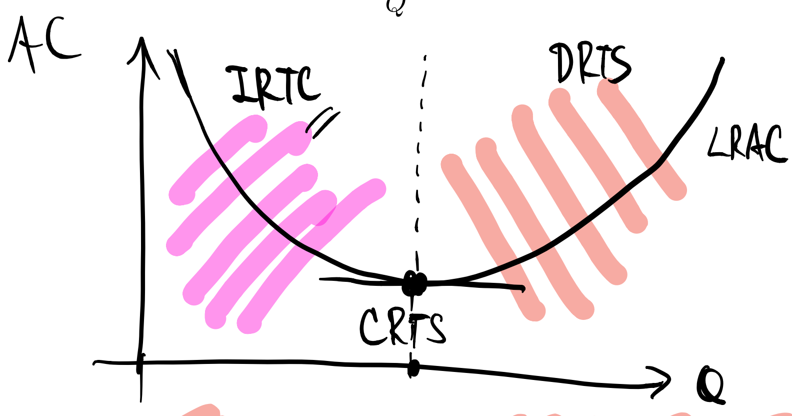
$$\{\lambda K_0, \lambda L_0\} \Rightarrow Q(\lambda K_0, \lambda L_0) = (\lambda K_0)^\alpha (\lambda L_0)^\beta = \lambda^{\alpha+\beta} K^\alpha L^\beta = \lambda^{\alpha+\beta} Q(K,L)$$

$$Q(\lambda K_0, \lambda L_0) = (\lambda \underbrace{K_0}_{K})^\alpha \cdot (\lambda \underbrace{L_0}_{L})^\beta$$
$$= \lambda^\alpha K_0^\alpha \lambda^\beta L_0^\beta$$
$$= \lambda^{\alpha+\beta} K_0^\alpha L_0^\beta$$
$$Q = \lambda^{\alpha+\beta} \cdot Q_0$$

$\alpha + \beta = 1$ CRTS.

$\alpha + \beta < 1$ DRTS
$\alpha + \beta = 1$ CRTS
$\alpha + \beta > 1$ IRTS

Recall $AC = \frac{TC}{Q}$



at IRTS, ↑ TC by 2, bit ↑ Q $> 2 \Rightarrow$ AC ↓

DRTS, ↑ TC by 2, bit ↑ Q $< 2 \Rightarrow$ AC ↑

AC ↓ : Q ↑ by 2 units $\Rightarrow$ TC ↑ by < 2 $\Rightarrow AC = \frac{TC\uparrow}{Q\uparrow\uparrow}$ ↓

AC ↑ : Q ↑ by 2 units $\Rightarrow$ TC ↑ by > 2 $\Rightarrow AC = \frac{TC\uparrow\uparrow}{Q\uparrow}$ ↑

$AC = \frac{2}{\sqrt{Q}}$: IRTS if $Q\uparrow$, $AC\downarrow$.

$AC(Q_0) = 2\cdot Q_0^{-\frac{1}{2}}$ suppose $Q_0 < Q_1$ ($Q\uparrow$)

$AC(Q_1) = 2\cdot Q_1^{-\frac{1}{2}}$ $AC(Q_0) = \frac{2}{\sqrt{Q_0}} > \frac{2}{\sqrt{Q_1}} = AC(Q_1)$

smaller ∵ denominator is larger.

ex $\boxed{TC = 2\sqrt{Q}}$

$$\text{AC} = \frac{TC}{Q} = \frac{2\sqrt{Q}}{Q} = \frac{2}{\sqrt{Q}} \quad \text{IRTS} \;\therefore Q\uparrow \Rightarrow AC\downarrow$$

$$\therefore F(\lambda x, \lambda y) = \lambda^r F(x,y) \quad \begin{array}{llcll} \text{if } r = 1 & , & CRTS & \Rightarrow & \text{AC constant} \\ r > 1 & , & \text{IRTS} & \Rightarrow & \text{AC} \downarrow \\ r < 1 & , & \text{DRTS} & \Rightarrow & \text{AC} \uparrow \end{array}$$

7.2 Types of Goods

ex.
$$\begin{aligned} U &= x + y \\ P_x &= 2 \\ P_y &= 1 \\ I &= 100 \end{aligned}$$

"Perfect substitute"

$100 - 2x - y = 0$

$1 - 2\lambda = 0 \Leftrightarrow \lambda = 0.5$

$1 - \lambda = 0 \Leftrightarrow \lambda = 1$

contradiction.

(1st approach): $\mathcal{L} = x + y + \lambda(100 - 2x - y)$

FONC

$$\left.\begin{aligned} \mathcal{L}_\lambda &= 100 - 2x - y &= 0 \\ \mathcal{L}_x &= 1 - 2\lambda &= 0 \\ \mathcal{L}_y &= 1 - \lambda &= 0 \end{aligned}\right\} \text{Lagragian cannot solve the corner solution.}$$

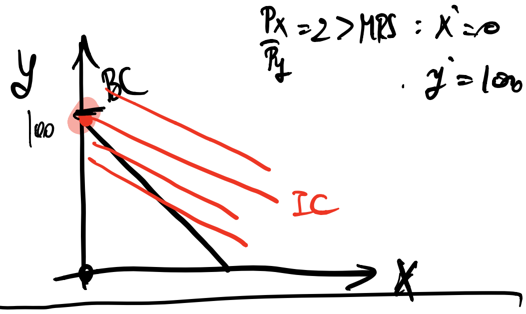
max $x+y$
st $2x + y = 100$.

(2nd approach)
$$\begin{aligned} MU_x &= 1 = \frac{\partial U}{\partial x} \\ MU_y &= 1 = \frac{\partial U}{\partial y} \\ MRS &= 1 \end{aligned}$$

∴ A consumer should buy more Y ∵ it is cheaper and give same MU

⇒ Demand for x = 0 and Demand for y = 100

$\frac{P_x}{P_y} = 2 > MRS$: $x^* = 0$
$y^* = 100$

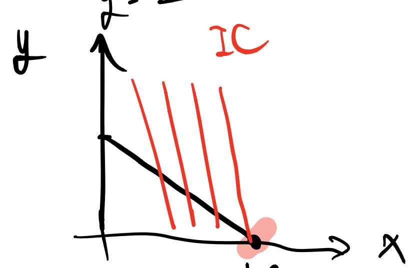


This case: $\frac{P_x}{P_y} = \frac{2}{1} = 2 > |MRS| = 1$

How about $|\frac{P_x}{P_y}| < |MRS|$

$|\frac{P_x}{P_y}| = |MRS|$

$\frac{P_x}{P_y} < MRS$. $P_x = 1$, $P_y = 2$ ⇒ $x^* = 100$, $y^* = 0$.



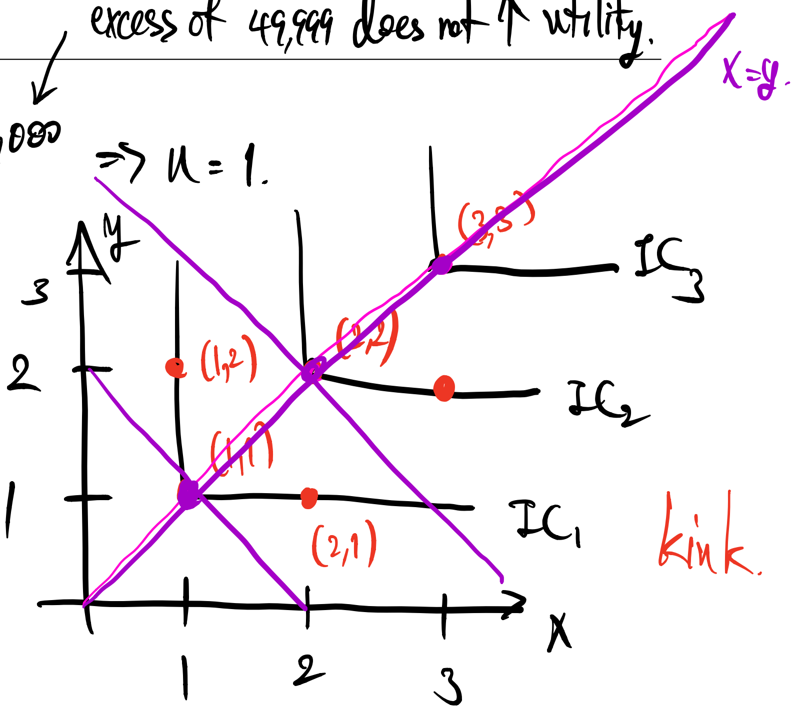
excess of 49,999 does not ↑ utility.

ex. U = min [x,y]

$X = 50,000$
$y = 1$
$\Rightarrow U = 1.$

x	y	u
1	1	1
1	2	1
2	1	1
2	3	2

"perfect complement"



ex. U = min [2x,y]

kink point is 2x = y

$(2,1) = U_1$ $(3,1) = U_1$ $(4,1) = U_1$
$(4,2) = U_2$
$(6,3) = U_3$
⋮

$$\text{ex.}\quad \left.\begin{aligned} U &= \min[x,y] \\ P_x &= 2 \\ P_y &= 1 \\ I &= 100 \end{aligned}\right\}$$

BC: $2x + y = 100$.

(1st approach) $\mathcal{L} = \min[x, y] + \lambda(100 - 2x - y)$

Solve for x^*, y^*

$$\left.\begin{aligned} \mathcal{L}_x &= \emptyset \\ \mathcal{L}_y &= \emptyset \\ \mathcal{L}_\lambda &= \emptyset \end{aligned}\right\}$$

(2nd approach)

kink point $\boxed{x = y}$ and budget constraint: $100 - 2x - y = 0$

$$\left.\begin{aligned} x &= y \\ 100 - 2x - y &= 0 \end{aligned}\right\} \qquad x^* = \frac{100}{3},\quad y^* = \frac{100}{3}.$$

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