6 Integration

6.1 Definition of Integral

For a function, $F(x)$ that its derivative is $f(x)$

Or mathematically $F'(x) = f(x)$

or an integral of $f(x)$

This function, $F(x)$, then can be called an **'anti-derivative'** of $f(x)$. For example,

- If $f(x) = nx^{n-1}$, then $F(x) = x^n$; as long as $n \neq 0$.
- If $f(x) = Ce^{Cx}$, then $F(x) = e^{Cx}$.
- If $f(x) = \frac{1}{x}$, then $F(x) = \ln x$.

It should be noted that a function does not have a **unique** anti-derivative.

For example, if $f(x) = 3x^2$

then $F(x) = x^3$ is one anti-derivative

or $F(x) = x^3 + 1$, $F(x) = x^3 + 8$, ... etc .... (Infinite number of anti-derivatives)

all there can be anti-derivative of $f(x)$

We may spot that any $F(x) = x^3 + C$ where C is a constant, can be anti-derivative of $f(x)$.

(You can try differentiate $F(x)$ and you will see that this is true because $\frac{d}{dx}(C) = 0$.)

$F'(x) = 3x^2 + \frac{dC}{dx} = 3x^2 = f(x)$ (doesn't matter what values of C are as long as C is a constant)

Hence, for this fuction $f(x) = 3x^2$,

$F(x) = x^3 + C$ where C is a constant and is called **a constant of integration**.

or the process of finding $F(x)$ where $F'(x) = f(x)$

This process of anti-differentiation is called **INTEGRATION**(n).
INTEGRATE (v.)

In other words, to integrate a function, $f(x)$ is to find $F(x)$ such that $F'(x) = f(x)$.

A symbol for integrating a function $f(x)$ is written as $\int f(x)dx$

$$\boxed{\int f(x)dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x)}$$

where $\int$ is called the *integral sign.*

f(*x*) is called the *integrand.*

C is called the *constant of integration.*

dx is part of the integral notation and indicates the variable involved.

REMEMBER that the integration is not unique; **DON'T FORGET THE CONSTANT OF INTEGRATION!**

6.2 Indefinite Integral

If we have to integrate a function, *f*(*x*) and values of *x* are not given, we said we have to integrate without a limit. Hence, the integral will be a function of *x*. (look like equation containing x)

= find indefinite integral

From some of the differentiation rules, the following integration formulas can be derived:

1	$\int k dx = kx + C$, k is a constant
2	$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$, $n \neq -1$
3	$\int e^x dx = e^x + C$
4	$\int \dfrac{1}{x} dx = \ln\lvert x\rvert + C$
5	$\int kf(x)dx = k\int f(x)dx$, k is a constant
6	$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

⟹ learn these (not given in exam)

NOTE: we can always check the integration result by differentiating it!

→ always do this to avoid making a mistake in exam.

Ex.1 Find the following integrals.

(i) $\int 5dx$ (use 1, $k=5$) [ANS: $5x + C$]

(ii) $\int x^4 dx$ (use 2, $n = 4 \Rightarrow \int x^4 dx = \dfrac{x^{4+1}}{4+1} + C$ [ANS: $\dfrac{x^5}{5} + C$]

$= \dfrac{x^5}{5} + C$

re ②, $n = -\frac{1}{2}$

(iii) $\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2\sqrt{x} + C$ [ANS: $2\sqrt{x}+C$]

(iv) $\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right) dx$ [ANS: $\frac{10}{9}x^{\frac{9}{5}} - \frac{7}{4}x^4 + 10e^x - x + C$]

in a solution in a separate file

(v) $\int \left(\frac{(x-1)(2x+7)}{3}\right) dx$ [ANS: $\frac{1}{3}\left[\frac{2}{3}x^3 + \frac{5}{2}x^2 - 7x\right] + C$]

in a solution in a separate file

Note that all answers in Ex1 are equations containing x which are expected when we do indefinite integral (or integrate without limits)

Ex.2: Suppose a firm has a marginal cost

$$MC = a + bQ + dQ^2$$

Find the equation of its total cost curve.

[Ans: $C(Q) = aQ + \frac{bQ^2}{2} + \frac{dQ^3}{3} + e$. e is the constant of integration. It is the value of $C(Q)$ when $Q = 0$, i.e. the fixed cost of production.]

> If a value of $f(x_i)$ [initial boundary condition] is given,
> the constant of integration (C) can be evaluated to obtain a specific equation.

Ex.3: If $f'(x) = 2x$ and $f(1) = 4$, find $f(x)$. [ANS: $x^2 + 3$]

$f(x) = 2\int x\,dx = \frac{2x^2}{2} + C = x^2 + C$, given $f(1) = 4$ $\therefore$ $4 = 1 + C$
$C = 3$

Therefore $f(x) = x^2 + 3$

Ex.4: Find the function $f(x)$ whose tangent has slope $\sqrt{x\sqrt{x\sqrt{x}}}$ for each value of x and

= saying $f'(x) = \sqrt{x\sqrt{x\sqrt{x}}}$

whose graph passes through the point (1,1). [ANS: $\frac{8}{15}x^{\frac{15}{8}} + \frac{7}{15}$]

= saying that $f(1) = 1$

$f'(x) = \sqrt{x\sqrt{x \cdot x^{1/2}}} = \sqrt{x \cdot x^{1/2} \cdot x^{1/4}} = \sqrt{x^{7/4}} = \left(x^{\frac{7}{4}}\right)^{\frac{1}{2}} = x^{\frac{7}{8}}$

$\int x^{7/8} dx = \frac{x^{\frac{7}{8}+1}}{\frac{7}{8}+1} + C = \frac{x^{15/8}}{\frac{15}{8}} + C = \frac{8}{15}x^{15/8} + C$ $\Longrightarrow$ given $f(1) = 1$ $\therefore$ $1 = \frac{8}{15} + C$ $\Rightarrow C = \frac{7}{15}$ $\therefore f(x) = \frac{8}{15}x^{15/8} + \frac{7}{15}$

Ex.5: For a firm, the marginal cost is $MC = 3Q + 4$. If the fixed cost are 40, find the total cost function $C(Q)$ [ANS: $C = \frac{3}{2}Q^2 + 4Q + 40$]

// $C(Q)$

$\rightarrow$ meaning $C(0) = 40$

$C(Q) = \int 3Q + 4 = \frac{3Q^2}{2} + 4Q + C$

given $C(0) = 40 \Rightarrow 40 = C$

$\therefore C(Q) = \frac{3}{2}Q^2 + 4Q + 40$

Again answers in EX3 – EX5 are equations containing variables (Q or x) because even a boundary condition is given, they are still indefinite integrals

✱✱

> If there are n numbers of constants of integration, n numbers of initial boundary conditions $f(x_i)$ are required to evaluate all constants of integration.

Ex.6: If $y'' = x^2 + x + 1$, find $y(x)$ for [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + C_1 x + C_2$]

$* \; y' = \int (x^2+x+1)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C_1$

$y = \int \left(\frac{x^3}{3} + \frac{x^2}{2} + x + C_1\right) dx = \frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + C_1 x + C_2$ ∴ need two boundary conditions

(i) when $y(0) = 0$ and $y(1) = \frac{7}{4}$ [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + x$]

$y(0) = 0 = C_2$

$y(1) = \frac{7}{4} = \frac{1}{12} + \frac{1}{6} + \frac{1}{2} + C_1 + 0$

$\therefore C_1 = \frac{7}{4} - \frac{3}{4} = \frac{4}{4} = 1$ $\quad \therefore y = \frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} + x$

(ii) when $y(0) = 0$ and $y'(0) = 0$ [ANS: $\frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2}$]

$y(0) = 0 = C_2$

use $y'(x)$ ✱

$y'(0) = 0 = \frac{0^3}{3} + \frac{0^2}{2} + 0 + C_1 \Rightarrow C_1 = 0$

$\therefore y = \frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2}$

NOTE that if $y'(a)$ and $y'(b)$ are given, these are not enough because C_2 cannot be determined.

If $y'(a)$ is given to be $= m$, then we can write

$m = \frac{a^3}{3} + \frac{a^2}{2} + a + C_1$ —— (1)

Similarly if $y'(b)$ is given to be n

$n = \frac{b^3}{3} + \frac{b^2}{2} + b + C_1$ —— (2)

both equations do not contain C_2 so they are not useful if we want to find C_2

Ex.7: Given the marginal-revenue function $MR = 2000 - 20Q - 3Q^2$. Find the demand function. [ANS: $2000 - 10Q - Q^2$]

Hint: If there is no product sold $(Q = 0)$ then the revenue is zero.

(Solution in a separate file)

6.3 Power Rule for Integration and Integration by Substitution.

This power rule is actually the same as 2 in 6.2

Again in order to integrate we must try to arrange the given derivative to look similar to one/or more of those formulas 1-6 in 6.2

$$\frac{d}{dx}\left(\frac{[u(x)]^{n+1}}{n+1}\right)=[u(x)]^n\cdot u'(x)$$

$$\therefore \quad \int[u(x)]^n\cdot u'dx=\frac{[u(x)]^{n+1}}{n+1}+C$$

But $\quad u'(x)=\frac{du}{dx}\Rightarrow u'(x)dx=du$

extra knowledge

I'm happy as long as you know how to use power rule in integration

In this case, we may have $\int[u(x)]^n dx \Rightarrow \therefore$ try to view/arrange it in the form of $\int u^n du \Rightarrow$ see example below for a better understanding

$$\boxed{\therefore \quad \int[u(x)]^n du=\frac{u^{n+1}}{n+1}+C}$$

Power Rule

> If u is differentiable and a function of x, then $\int u^n du=\frac{u^{n+1}}{n+1}+C$ if $n\neq -1$.

$\rightarrow$ similar to rule 2 in 6.2

Ex.8: Find $\int(x+5)^5 dx$. $\Rightarrow$ try to make this look like 2 in 6.2 by substitute $x+5=u$

$\int x^n dx=\frac{x^{n+1}}{n+1}$

only change $x=u$, $dx=du$

Integration by substitution, try $\quad u=x+5\Rightarrow du=1\cdot dx$

or $\frac{du}{dx}=1 \Rightarrow du=1\cdot dx$

$$\int(x+5)^5 dx=\int u^5 du=\frac{u^6}{6}+C$$

Substitute u gives $\quad \int(x+5)^5 dx=\frac{(x+5)^6}{6}+C$

Note that you **MUST** give the answer explicitly in term or x **NOT** u.

Ex.9: Find $\int\sqrt[3]{6x}dx$. $\quad$ (**Hint:** u=6x) $\quad$ [ANS: $\frac{(6x)^{\frac{4}{3}}}{8}+C$]

$du=6\,dx$

$\therefore dx=\frac{du}{6}$ — (1)

$$\int u^{1/3}dx=\int u^{1/3}\frac{du}{6}=\frac{u^{\frac{1}{3}+1}}{(\frac{1}{3}+1)6}+C=\frac{u^{4/3}}{8}+C=\frac{(6x)^{4/3}}{8}+C$$

need to change dx to du before integrate by using $dx=\frac{du}{6}$ in (1)

Ex.10: Show that $\int x^2(3x^3+7)^3dx=\frac{(3x^3+7)^4}{36}+C$ $\quad$ [**Hint:** $u=3x^3+7$]

$\int x^2u^3dx \Rightarrow$ here we have to change x^2 to be an expression in terms of u and change dx to something in terms of du

As $u=3x^3+7 \Rightarrow du=9x^2dx$

$\frac{u-7}{3}=x^3$

$\left(\frac{u-7}{3}\right)^{\frac{1}{3}}=x$

$\therefore x^2=\left(\frac{u-7}{3}\right)^{2/3}$ — (1)

$dx=\frac{1}{9x^2}du$

in terms of x^2 which will cancel out x^2 in $\int x^2u^3dx$

Luckily don't need (1)

$$\therefore \int x^2u^3dx=\int\frac{\cancel{x^2}u^3}{9\cancel{x^2}}du=\int\frac{u^3}{9}du=\frac{u^4}{36}+C=\frac{(3x^3+7)^4}{36}+C$$

6.4 More Integration by Substitution

Let u be a function of x,

$$\int e^u du = e^u + C$$

$$\int \frac{1}{u} du = \ln|u| + C$$

Ex.11: Show that $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$. **[Hint:** $u = ax$ **]**

$u = ax \Rightarrow du = a\,dx$

$\therefore dx = \frac{du}{a}$

$\int e^u dx = \int \frac{e^u}{a} du = \frac{e^u}{a} + C$

$= \frac{1}{a} e^{ax} + C$

solutions in separate file.

Ex.12: Show that $\int a^x dx = \frac{1}{\ln a} a^x + C$.

Ex.13: Show that $\int \left((2t+1)e^{t^2+t} dt\right) = e^{t^2+t} + C$. **[Hint:** $u = t^2 + t$ **]**

Ex.14: Show that $\int \frac{7t}{5t^2-6} dt = \frac{7}{10} \ln|5t^2 - 6| + C$.

Fill in the table by specifying the substitution you may choose for each integral.

Integral	Substitution u
1. $\int (3x+4)^{\frac{5}{2}} dx$	
2. $\int \left(\frac{4}{3-x}\right) dx$	
3. $\int te^{2-t^2} dt$	
4. $\int t(2+t^2)^3 dt$	
5. $\int \frac{3}{(2x-5)^4} dx$	
6. $\int x^2 e^{-x^3} dx$	
7. $\int \frac{e^t}{e^t+1} dt$	
8. $\int \frac{t+3}{\sqrt[3]{t^2+6t+5}} dt$	

6.4.1 Integration Hints

- Try to arrange the integrand into a familiar integration forms (seen formulas). If you can't find the right form, try some more! There is no shortcut! **ONLY PRACTICE!**
- When integrating fraction forms, sometimes a preliminary long division is needed to get familiar integration forms. understan this through Ex.15, Ex.16, Ex.17

Ex.15: Show that $\int\left(\frac{x+3}{x+6}\right)dx = x-3\ln|x+6|+C$.

$\int \frac{x+3}{x+6}$ is problematic because if substitute $u = x+6$, we still have $x+3$ on the top of the fraction. For something like this

Try to arrange $\frac{x+3}{x+6} = m \pm \frac{C}{x+6}$ where $\overset{mx}{C}$ = constants. In this case $\frac{x+3}{x+6} < 1$

$\therefore \frac{x+3}{x+6} = 1 - \frac{C}{x+6} = \frac{x+6}{x+6} - \frac{C}{x+6} \Rightarrow C = 3 \therefore \int\left(1-\frac{3}{x+6}\right)dx = x - 3\ln(x+6) + C$

Ex.16: Show that $\int\left(\frac{x^3+x^2-x-3}{x^2-3}\right)dx = \frac{x^2}{2}+x+\ln|x^2-3|+C$.

Ex.17: Show that $\int\left(\frac{9x^2+5}{3x}\right)dx = \frac{3}{2}x^2+\frac{5}{3}\ln|x|+C$.

HW **Ex.18:** Show that $\int\left(\frac{6x^2-11x+5}{3x-1}\right)dx = x^2-3x+\frac{2}{3}\ln|3x-1|+C$.

HW **Ex.19:** Show that $\int 2x(x^2+3)^5\,dx = \frac{(x^2+3)^6}{6}+C$.

no fractions, use substitution technique

HW **Ex.20:** Show that $\int x^3 e^{4x^4}\,dx = \frac{1}{16}e^{4x^4}+C$.

Ex.21: $y' = (3-2x)^2$, $y(0)=1$, $y(x)=?$. [ANS: $-\frac{1}{6}(3-2x)^3+\frac{11}{2}$]

Ex.22: $y'' = \frac{1}{x^2}$, $y'(-1)=1$, $y(1)=0$, $y(x)=?$. [ANS: $-\ln|x|$]

In conclusion from EX. 15 – EX. 18, if the integrals contain fractions

$\frac{m(x)}{n(x)} \Rightarrow$ if the power of $m(x)$ equals or larger than $n(x)$

You have to try to divide $m(x)$ with $n(x)$ to get $o(x) + \frac{C}{n(x)}$ before doing any integration.

6.5 Summation (Revisions)

$$\sum_{i=0}^{n} i = 0+1+2+.....+n = \sum_{k=0}^{n} k$$

$\sum$ = notation for summation called 'sigma'.

i, k = index of summation (a 'dummy' symbol)

$0, n$ = limits of summation

If $n = 5$, $\sum_{i=0}^{5} i = 0+1+2+3+4+5$

Ex.23: Show that $\sum_{i=1}^{5} 3i^2 = 3\sum_{i=0}^{5} i^2$.

ANS:
$$\begin{aligned}\sum_{i=1}^{5} 3i^2 &= 3(1)^2 + 3(2)^2 + 3(3)^2 + 3(4)^2 + 3(5)^2 \\ &= 3\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2\right) \\ &= 3\sum_{i=0}^{5} i^2\end{aligned}$$

$$\boxed{\sum_{i=0}^{n} Cx_i = C\sum_{i=0}^{n} x_i}$$

Ex.24: If $\sum_{k=1}^{n} k = 1+2+3+....+n$, show that $\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$ is the general form in term of n.

ANS: Let $\quad S = \sum_{k=1}^{n} k = 1+2+3...+n \qquad (1)$

Reverse $\quad S = n+(n-1)+(n-2)...+1 \qquad (2)$

(1)+(2) $\quad 2S = (n+1)+(n-1+2)+(n-2+3)+...+(n+1)$

$$2S = (n+1)+(n+1)+(n+1)+...+(n+1)$$

$$2S = n\cdot(n+1)$$

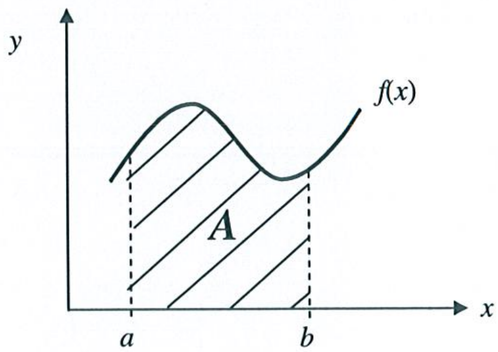
$$S = \frac{n\cdot(n+1)}{2} = \sum_{k=1}^{n} k$$

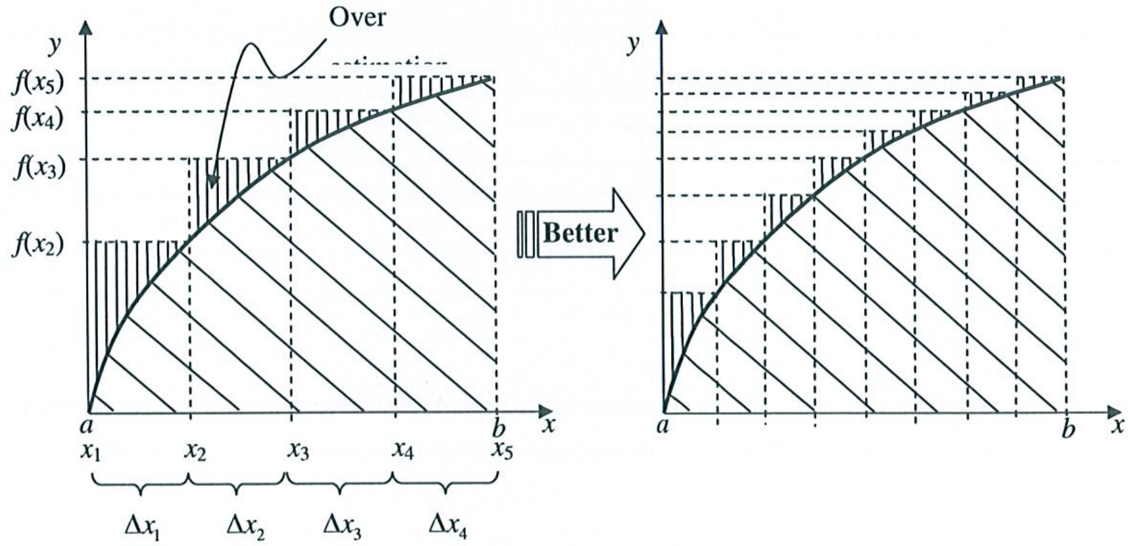
Ex.25: Find $\sum_{i=1}^{5} i$ and $\sum_{i=1}^{100} i$. [ANS: 15, 5050]

6.6 Definite Integral as an Area under a Curve

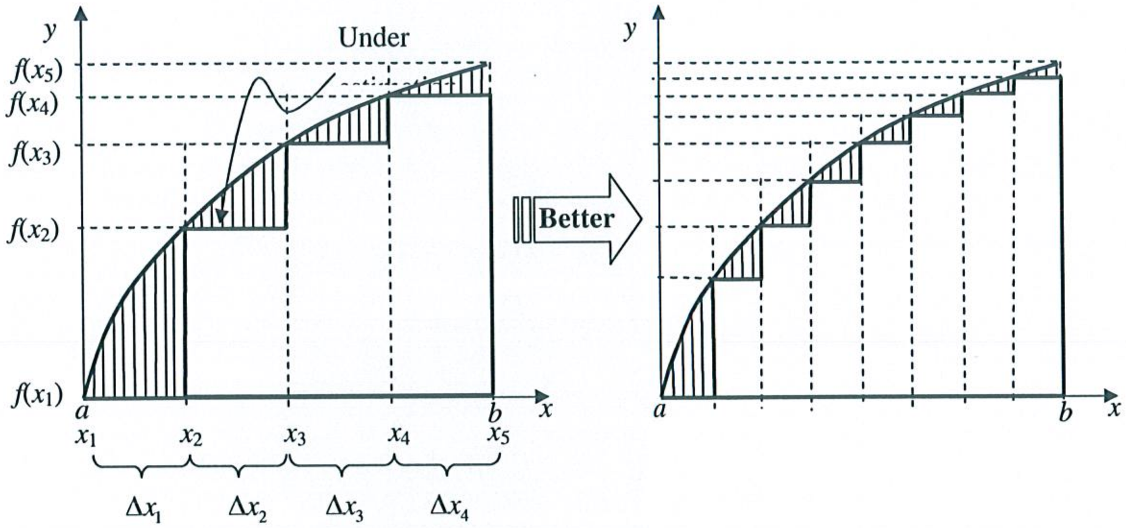
Area under a curve can be expressed as a limit of a sum of terms called a **definite integral.**

$$A = \int_a^b f(x)dx = \text{Area under a curve between } x = [a, b]$$

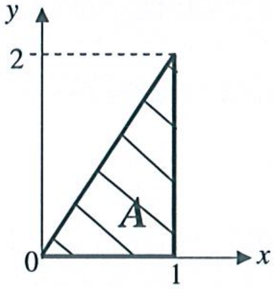




$$A^* = f(x_2)\Delta x_1 + f(x_3)\Delta x_2 + f(x_4)\Delta x_3 + f(x_5)\Delta x_4 = \sum_{i=1}^{4} f(x_{i+1})\Delta x_i$$

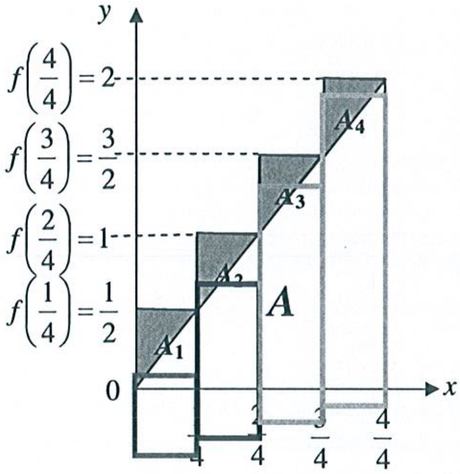


$$A^{*} = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + f(x_4)\Delta x_4 = \sum_{i=1}^{4} f(x_i)\Delta x_i$$



$f(x) = 2x$

$A = \frac{1}{2} \cdot 1 \cdot 2 = 1$



$\boxed{A_1 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}}$

$\boxed{A_2 = \frac{1}{4} \times 1 = \frac{1}{4}}$

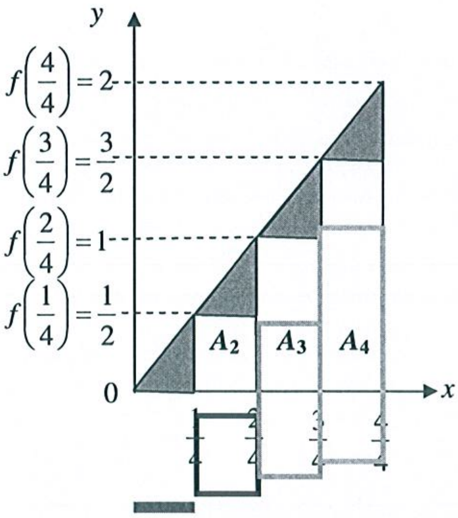
$\boxed{A_3 = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}}$

$\boxed{A_4 = \frac{1}{4} \times 2 = \frac{1}{2}}$

Estimated area (Over estimate)

$$\overline{S}_4 = A_1 + A_2 + A_3 + A_4 = \left(\frac{1+2+3+4}{8}\right) = \frac{5}{4}$$

Over estimate by $= \frac{5}{4} - 1 = \frac{1}{4}$



$\boxed{A_1 = 0}$

$\boxed{A_2 = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}}$

$\boxed{A_3 = \frac{1}{4} \times 1 = \frac{1}{4}}$

$\boxed{A_4 = \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}}$

Estimated area (Under estimate)

$$\underline{S}_4 = A_1 + A_2 + A_3 + A_4 = \left(\frac{0+1+2+3}{8}\right) = \frac{3}{4}$$

Under estimate by $= 1 - \frac{3}{4} = \frac{1}{4}$

$$y = f(x) = 2x, \qquad x = [0,1], \qquad \Delta x = \frac{1}{4}$$

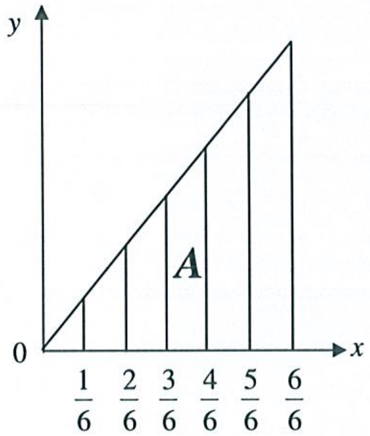
$$\overline{S}_4 = \frac{1}{4}f\left(\frac{1}{4}\right)+\frac{1}{4}f\left(\frac{2}{4}\right)+\frac{1}{4}f\left(\frac{3}{4}\right)+\frac{1}{4}f\left(\frac{4}{4}\right) \qquad \underline{S}_4 = \frac{1}{4}f(0)+\frac{1}{4}f\left(\frac{1}{4}\right)+\frac{1}{4}f\left(\frac{2}{4}\right)+\frac{1}{4}f\left(\frac{3}{4}\right)$$

$$\overline{S}_4 = \frac{1}{4}\left[2\left(\frac{1}{4}\right)+2\left(\frac{2}{4}\right)+2\left(\frac{3}{4}\right)+2\left(\frac{4}{4}\right)\right] \qquad \underline{S}_4 = \frac{1}{4}\left[2(0)+2\left(\frac{1}{4}\right)+2\left(\frac{2}{4}\right)+2\left(\frac{3}{4}\right)\right]$$

$$\overline{S}_4 = \frac{5}{4} > 1 \quad \text{(Exact area = 1)} \qquad \underline{S}_4 = \frac{3}{4} < 1 \quad \text{(Exact area = 1)}$$

$$\overline{S}_4 = \sum_{i=1}^{4} f(x_i)\Delta x$$

$$\underline{S}_4 = \sum_{i=1}^{3} f(x_i)\Delta x$$



$$f(x) = 2x$$

$$x = [0,1]$$

$$\Delta x = \frac{1}{6}$$

$$\overline{S}_6 = \sum_{i=1}^{6} f(x_i)\Delta x$$

$$\underline{S}_6 = \sum_{i=1}^{5} f(x_i)\Delta x$$

$$\overline{S}_6 = \frac{1}{6}f\left(\frac{1}{6}\right)+\frac{1}{6}f\left(\frac{2}{6}\right)+\frac{1}{6}f\left(\frac{3}{6}\right)+\frac{1}{6}f\left(\frac{4}{6}\right)+\frac{1}{6}f\left(\frac{5}{6}\right)+\frac{1}{6}f\left(\frac{6}{6}\right)$$

$$\overline{S}_6 = \frac{1}{6}\left[2\left(\frac{1}{6}\right)+2\left(\frac{2}{6}\right)+2\left(\frac{3}{6}\right)+2\left(\frac{4}{6}\right)+2\left(\frac{5}{6}\right)+2\left(\frac{6}{6}\right)\right]$$

$$\overline{S}_6 = \frac{1}{18}[1+2+3+4+5+6] = \frac{21}{18} = \frac{7}{6} \qquad \text{Over estimate by } = \frac{7}{6} - 1 = \frac{1}{6}$$

$$\underline{S}_6 = \frac{1}{6}f(0)+\frac{1}{6}f\left(\frac{1}{6}\right)+\frac{1}{6}f\left(\frac{2}{6}\right)+\frac{1}{6}f\left(\frac{3}{6}\right)+\frac{1}{6}f\left(\frac{4}{6}\right)+\frac{1}{6}f\left(\frac{5}{6}\right)$$

$$\underline{S}_6 = \frac{1}{6}\left[2(0)+2\left(\frac{1}{6}\right)+2\left(\frac{2}{6}\right)+2\left(\frac{3}{6}\right)+2\left(\frac{4}{6}\right)+2\left(\frac{5}{6}\right)\right]$$

$$\underline{S}_6 = \frac{1}{18}[0+1+2+3+4+5] = \frac{15}{18} = \frac{5}{6} \qquad \text{Under estimate by } = 1 - \frac{5}{6} = \frac{1}{6}$$

The estimated result is getting closer as the interval (Δx) gets **SMALLER!**

General Term $f(x) = 2x$ Divide $x = [0,1]$ into n subintervals. Hence, $\Delta x = \frac{1}{n}$

$$\overline{S}_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\overline{S}_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \ldots + \frac{1}{n} f\left(\frac{n}{n}\right)$$

$$\overline{S}_n = \frac{1}{n}\left[2\left(\frac{1}{n}\right) + 2\left(\frac{2}{n}\right) + 2\left(\frac{3}{n}\right) + \ldots + 2\left(\frac{n}{n}\right)\right]$$

$$\overline{S}_n = \frac{2}{n^2} \cdot [1 + 2 + 3 + \ldots + n]$$

$$\overline{S}_n = \frac{2}{n^2} \cdot \left[\frac{n(n+1)}{2}\right] = \frac{n+1}{n}$$

Note: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$\underline{S}_n = \sum_{i=1}^{n-1} f(x_i)\Delta x$$

$$\underline{S}_n = \frac{1}{n} f\left(\frac{0}{n}\right) + \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \ldots + \frac{1}{n} f\left(\frac{n-1}{n}\right)$$

$$\underline{S}_n = \frac{1}{n}\left[2\left(\frac{0}{n}\right) + 2\left(\frac{1}{n}\right) + 2\left(\frac{2}{n}\right) + \ldots + 2\left(\frac{n-1}{n}\right)\right]$$

$$\underline{S}_n = \frac{2}{n^2} \cdot [0 + 1 + 2 + \ldots + (n-1)]$$

$$\underline{S}_n = \frac{2}{n^2} \cdot \left[\frac{(n-1)n}{2}\right] = \frac{n-1}{n}$$

Note: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

As subinterval is getting smaller ($\Delta x \to 0$) ➔ n is larger ➔ $\overline{S}_n, \underline{S}_n$ closer to the exact area A.

$$\overline{S}_n = \frac{n+1}{n} \qquad \underline{S}_n = \frac{n-1}{n}$$

Question: For $\overline{S}_n = \dfrac{n+1}{n}$ and $\underline{S}_n = \dfrac{n-1}{n}$, what happen if $n \to \infty$?

$$\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

$$\lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$$

$$\therefore \quad \lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \underline{S}_n = 1$$

As $n \to \infty$, $\quad \overline{S}_n \to A \quad$ and $\quad \underline{S}_n \to A \quad$ (A = 1, exact area)

$$\underline{S}_n \le A \le \overline{S}_n$$

Common limit of $\underline{S}_n$ and $\overline{S}_n$ ➔ Definite integral of $f(x)$

$$\therefore A = \int_0^1 2x\,dx = 1$$

The **common** limit of $\underline{S}_n$ and $\overline{S}_n$ as $n \to \infty$, if it exists, is called the **definite integral** of $f(x)$ over (a,b). a & b are limits

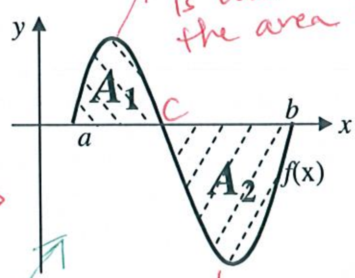
$$A = \int_a^b f(x)\,dx$$

For example,

$$\int_0^1 2x\,dx = \left[x^2\right]_0^1 = 1^2 - 0^2 = 1 = \lim_{n\to\infty} \underline{S}_n = \lim_{n\to\infty} \overline{S}_n$$

All I would like you to accept for now is that

$A = \int_a^b f(x)\,dx$ = areas $A_1 + A_2$



NOTE: Definite integral **DOES NOT** always represents an area.

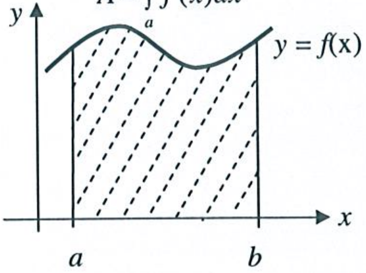
$\int_a^b f(x)\,dx = A_1 - A_2 =$ **NEGATIVE.** Area can't be negative!

But if you want to find an area, you have to spot whether graph of a function crosses x-axis at any x-values between the limit a & b. For example, in the picture, to find an area bounded by f(x) & x-axis between x = a to x = b, you have to find A_1 which is $\int_a^c f(x)\,dx$ then find A_2 which is $\left|\int_c^b f(x)\,dx\right|$ and sum them together.

6.7 *Fundamental Theorem of Integration Calculus*

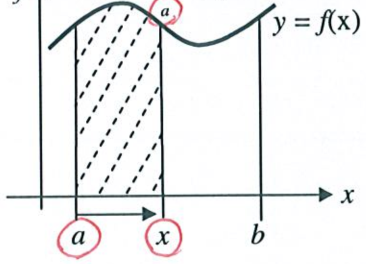
- To informally develop the fundamental theorem of integral calculus and to use it to complete definite integrals.
- To obtain a change in function values when the rate of change of the function is known.

$A = \int_a^b f(x)dx$



A = Area between the curve $y = f(x)$, $x = a$, $x = b$, and the x axis. = number

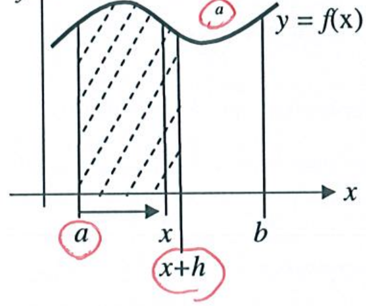
$A(x) = \int_a^x f(x)dx$



A = Area function

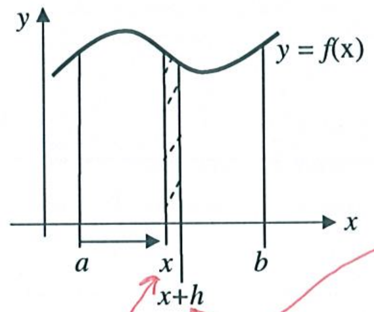
$A(x)$ = Area between the curve $y = f(x)$, $x = x$, $x = a$, and the x axis. = function containing x

$A(x+h) = \int_a^{x+h} f(x)dx$



Increase x by h unit

$A(x+h)$ = Area between $y = f(x)$, $x = (x+h)$, $x = a$, and the x axis.



The different in area

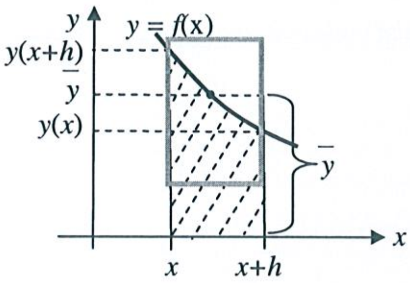
$\Delta A = A(x+h) - A(x)$

$= \int_a^{x+h} f(x)dx - \int_a^{x} f(x)dx$

$= \int_x^{x+h} f(x)dx$

So if the limits are from a to b e.g. $\int_a^b f(x)\,dx$

Then it is the area under curve when $x = a$ to $x = b$



As $h \to 0$, area [hatched region] is approximately rectangle.

Estimate to rectangle with height $\bar{y}$

$$\Delta A = A(x+h) - A(x) = \bar{y} \cdot h$$

$$\bar{y} = \frac{A(x+h) - A(x)}{h}$$

$$\bar{y} = \frac{A(x+h) - A(x)}{h}$$

$h \to 0$ and $\bar{y} \to f(x)$

$$\therefore \quad \lim_{h\to 0}\left[\frac{A(x+h) - A(x)}{h}\right] = f(x)$$

This is the definition of a derivative of A.

$$\therefore \quad A'(x) = f(x)$$

A **derivative** of an area A is $f(x)$ the function itself.

A is an **anti-derivative** of $f(x)$.

$$\therefore \quad \int f(x)dx = A(x) + C_1 \qquad (1)$$

The integral of a function $f(x)$ is the area function $A(x)$. Anti-derivative of $f(x)$ is $F(x)$.

$$\therefore \quad \int f(x)dx = F(x) + C_2 \qquad (2)$$

(1) = (2), $\quad A(x) + C_1 = F(x) + C_2$

$$A(x) = F(x) + C \qquad (3)$$

$$A(0) = 0 = F(a) + C \Rightarrow C = -F(a)$$

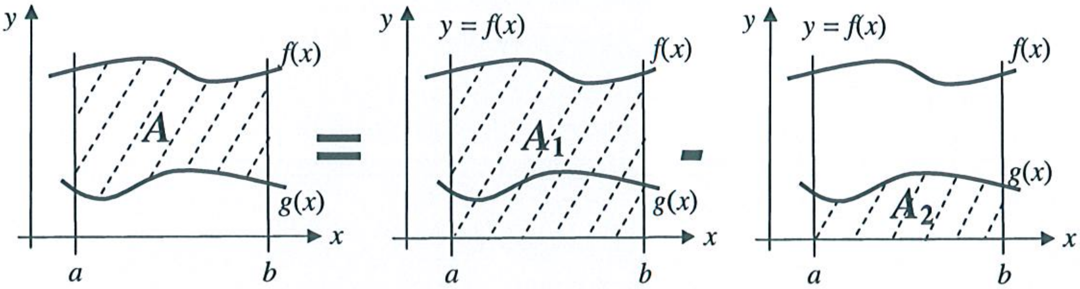
(3) becomes $\quad A(x) = F(x) - F(a) \qquad (4)$

If $x = b$, (4) becomes

$$A(x) = F(b) - F(a) = \int_a^b f(x)dx$$

Hence, if f is continuous on interval $[a,b]$, $F(x)$ is any anti-derivative of $f(x)$ on the interval then

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$ ** learn this (not given in exam)



6.7.1 Area between Curves

Find area bounded by $f(x)$, $g(x)$, $x = a$, and $x = b$.

$$A = A_1 - A_2$$

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$A = \int_a^b [f(x) - g(x)]dx$$

$$\boxed{\therefore \quad \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx}$$

Similarly others properties of definite integral can be derived from area between curves.

6.7.2 Properties of Definite Integral

→ given limits a & b

1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2) $\int_a^a f(x)dx = 0$

3) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (k is a constant)

4) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5) $\int_a^b f(x)dx = -\int_b^a f(t)dt$ Any variable will give the same result.

6) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

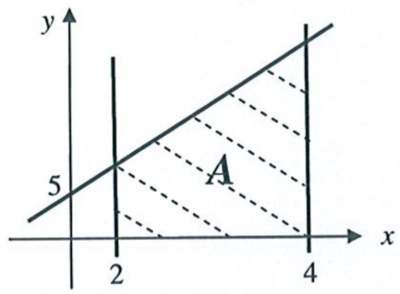
learn these (not given in exam)

But they are some ways similar to 6.2

Ex.26: Use a definite integral to find the area of the region bounded by the given curve, the x axis, and the given lines.

(Note: We need to sketch the curve to make sure that there will be no negative area!)

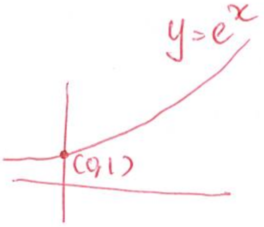
(i) $y = x + 5$, $x = 2$, and $x = 4$



$$A = \int_2^4 (x+5)dx = \left[\frac{x^2}{2} + 5x\right]_2^4$$

$$A = \left(\frac{4^2}{2} + 5(4)\right) - \left(\frac{2^2}{2} + 5(2)\right)$$

$$A = 28 - 12 = 16$$



(ii) $y = e^x$, $x = 1$, and $x = 3$ $A = \int_1^3 e^x dx = \left[e^x\right]_1^3 = e^3 - e$ [Ans: $e^3 - e$]

(iii) $y = x^2 - 2x$, $x = 1$, and $x = 3$ [Ans: 2]

HW

(iv) $y = x^3$, $x = -2$, and $x = 4$ [ANS: 68]

(v) $y = \dfrac{1}{x}$, $x = 1$, and $x = e$ [ANS: 1]

(vi) $y = \dfrac{1}{x}$, $x = 1$, and $x = e^2$ [ANS: 2]

(iii) $y = x^2 - 2x = 0 = x(x-2) \Rightarrow x = 0, 2$

$y(1) = 1^2 - 2 = -1$ $\quad y(3) = 9 - 6 = 3$

$\therefore$ from $x = 1$ to $x = 2$, the area bounded by the curve is negative (the graph of function is below x-axis)

whereas when $x = 2$ to $x = 3$, the area is positive

$\therefore$ To find the area bounded we need to find $A_1 = \left|\int_1^2 (x^2 - 2x)dx\right|$

and $A_2 = \int_2^3 (x^2 - 2x)dx$ and the total area is $A_1 + A_2$

This is ⊖ confirming negative area

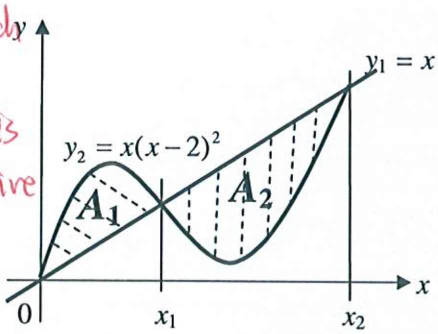
$$A_1 = -\int_1^2 x^2 - 2x\,dx = -\left[\frac{x^3}{3} - x^2\right]_1^2 = -\left(\frac{8}{3} - 4 - \left(\frac{1}{3} - 1\right)\right) = -\frac{8 - 12 - 1 + 3}{3} = \frac{2}{3}$$

$$A_2 = \left[\frac{x^3}{3} - x^2\right]_2^3 = \frac{27}{3} - \frac{27}{3} - \left(\frac{8}{3} - 4\right) = 0 - \frac{8}{3} + \frac{12}{3} = \frac{4}{3}$$

$$A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$$

Ex.27: Express the area in terms of integrals

Note we can sketch a graph to see where the area is positive or negative



Find intersection points

$x(x-2)^2 = x$

$(x-2)^2 = 1$ or $x = 0$

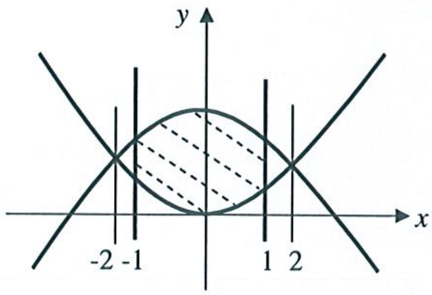
$x - 2 = \pm 1 \Rightarrow x = 3, 2, 0$

$$A = \int_0^2 x(x-2)^2\,dx - \int_0^2 x\,dx + \int_2^3 x\,dx - \int_2^3 x(x-2)^2\,dx$$

In this case the area bounded is given as integrals of $y_2 - y_1$ $\therefore A_1$ where $y_2 > y_1$ is considered as a positive area. However if the area bounded is found as integrals of $y_1 - y_2$, then A_1 is $\ominus$. Both methods should give the same answer anyway.

Ex.28: Find the area of the region bounded by the graphs of the given equations. Be sure to find any needed points of intersection.

(i) $y = 8 - x^2$, $y = x^2$, $x = -1$, and $x = 1$ [Ans: 44/3]



Intersection: $8 - x^2 = x^2$

$2x^2 = 8 \Rightarrow x = \pm 2$

$$A = \int_{-1}^{1} \left[\left(8 - x^2\right) - x^2\right] dx = \frac{44}{3}$$

$\therefore -1$ to 1 are within the positive area

(ii) $y = 2 - x^2$, $y = x$ [Ans: $\int_{-2}^{1} \left[\left(2 - x^2\right) - x\right] dx = \frac{9}{2}$]

HW (iii) $y = x^2 + 2$, $y = 8$ [Ans: $\int_{-\sqrt{6}}^{\sqrt{6}} \left[8 - \left(2 - x^2\right)\right] dx = 8\sqrt{6}$]

6.7.3 The Average Value of a Function

Let $f(x)$ be a function that is continuous on the interval $a \le x \le b$. Then the ***average value*** V of $f(x)$ over $a \le x \le b$ is given by the definite intergral

$$V = \frac{1}{b-a}\int_a^b f(x)dx$$

Ex.29: A manufacture determines that t moths after introducing a new product, the company's sales will be $S(t)$ thousand dollars, where

$$S(t) = \frac{750t}{\sqrt{4t^2+25}}$$

What are the average monthly sales of the company over **the first 6 months** after the introduction of the new product? [Ans: 250]

$$\bar{S} = \frac{1}{6-0}\int_0^6 \frac{750t}{\sqrt{4t^2+25}}\,dt = \frac{1}{6}\int_0^6 \frac{750t}{\sqrt{4t^2+25}}\,dt$$

This is a fraction but if diff $4t^2+25 \Rightarrow$ get $8t$ = constant $\times$ t similar to the top of the fraction. Hence can use substitution $u = 4t^2+25$

$\therefore\; du = 8t\,dt \Rightarrow dt = \frac{1}{8t}du$

169 → This is u when t=6

$$\bar{S} = \frac{1}{6}\int_0^6 750\cancel{t}\,u^{-1/2}\cdot\frac{1}{8\cancel{t}}\,du = \frac{750}{48}\int_{25}^{169} u^{-1/2}du$$

25 → This is u when t=0

$$= \frac{750}{48}\left[\frac{u^{1/2}}{1/2}\right]_{25}^{169} = \frac{750}{48}\left[26-10\right] = \underline{250}$$

6.8 Definite Integral (With numerical limit, the answer is real number.)

Unlike indefinite integral ⇒ no limit ⇒ the answer is an equation

If $F(x)$ is any anti-derivative of the function $f(x)$, the 'definite' integral of $f(x)$ between a and b, $(a \le x \le b)$, is

$$\boxed{\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b \qquad a < b}$$

a is the lower limit of integration and *b* is the upper limit of integration. **Definite** integral will give a **definite** numerical value, **NO *x* or *C*.** *C* will cancel itself out during the calculation.

e.g. $\int f(x)\,dx = F(x) + C$ where C = constant

$\therefore$ when $x = b \Rightarrow$ integral $= F(b) + C$

when $x = a \Rightarrow$ integral $= F(a) + C$

The difference in area $= F(b) - F(a)$

no need to find C

from $x=1$ to $x=5$, graph is always above x-axis
∴ can find integration straightaway

Ex.30: Show that $\int_1^5 3x^2 dx = 124$ and $\int_0^2 (4-x^2)\,dx = \frac{16}{3}$. ⇒ between $x=0$ to $x=2$ graph of function is above x-axis ⇒ can find integration without splitting between positive & negative areas.

$\int_1^5 3x^2 = \left[\frac{3x^3}{3}\right]_1^5 = 5^3 - 1 = 124$

$\int_0^2 (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$

When using integration by substitution for a definite integral, we need to **substitute the limit** as well or **substitute before taking limits.**

like in Ex. 29 where limit of $u = 25$ & 169 are used instead.

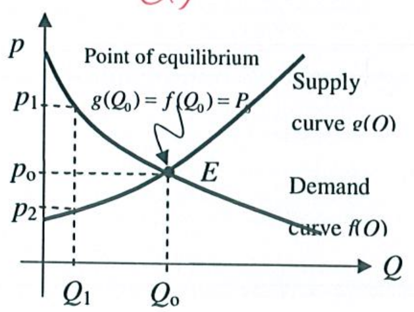
HW **Ex. 31:** Show that $\int_1^2 (2x^3-1)^2(6x^2)\,dx = \frac{3374}{3}$.

Ex. 32: Show that $\int_9^{64} \left(\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\right) dx = \frac{76}{3}$.

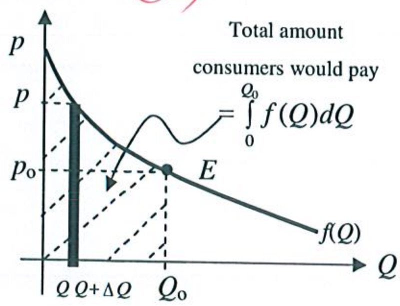
6.9 Consumers' and Producers' Surplus

Economists are interested in studying how consumers' and producers' welfare are influenced by changes in the economic parameters. A common measure of such welfare changes is the consumers' and producers' surplus.

(C1)



(C2)



p = price **per unit (example of unit is bahts/unit)** Q = Quntity buy or sell

- Supply curve $[g(Q)]$ indicates the price p per unit at which the **producers** will **sell** (or **supply**) Q units.
- Demand curve $[f(Q)]$ indicates the price p per unit at which the **consumers** will **purchase** (or **demand**) Q units.

- **Point of equilibrium** (E) indicates the consumers to purchase precisely the same amount (Q_0) that the producers are willing to offer at the price per unit (p_0).

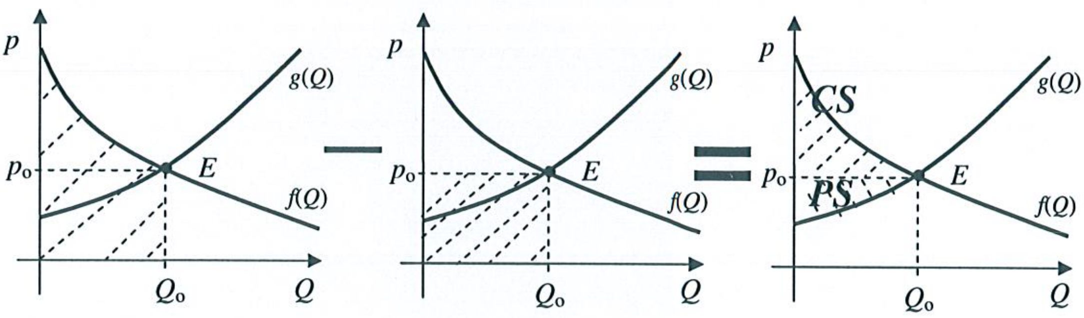
up to here is a revision

Demand and supply curves are intercepted. $p_0 = q(Q_0) = f(Q_0)$

- **If the market is at equilibrium** $(p = p_0)$, there are consumers who are willing to **pay more** than p_0. For example, at the price of p_1, consumers would buy Q_1 unit. These consumers **benefiting** from the lower price p_2. on previous page which is the price by supplier at Q_1

Figure ② on previous page

- $p\Delta Q$ is the total amount of money that customers would spend by buying ΔQ unit if the price per unit was p. In fact, the price is at the equilibrium p_0 so the customers only buy $p_0\Delta Q$. Thus, the customers ***save*** or ***benefit*** $(p - p_0)\Delta Q$.
- Area under the curve is the amount of money to sell or buy the product.



$$\int_0^{Q_0} f(Q)dQ \quad - \quad \int_0^{Q_0} P_0 dQ \quad = \quad \int_0^{Q_0}[f(Q) - P_0]dQ = CS$$

- For those who are willing to pay for commodity at price p_0 or higher, the total amount they are willing to pay is the total area $\int_0^{Q_0} f(Q)dQ$. In fact, consumers only pay $\int_0^{Q_0} P_0 dQ$. Thus, the customers ***save***

$$CS = \int_0^{Q_0}[f(Q) - P_0]\,dQ.$$

- **Consumers' Surplus [*CS*]** is the total amount ***save*** by all the customers buying he commodity at a price lower than what they are maximally are willing to pay.
- Some producers are also benefiting from the equilibrium price, since they are willing to supply at prices ***lower*** than p_0. Similarly,

$$PS = \int_0^{Q_0}[P_0 - f(Q)]\,dQ$$

- **Producers' Surplus [*PS*]** is the total benefit of all the producers who obtain a price higher than the price they are willing to sell it for.

This is actually a revision // $q(p)$: q as a function of p

Ex.33: A demand function is $q = 10000e^{-0.02p}$ unit. At what price will revenue be maximised? $R(q) = P(q)q$ OR $q(p) \cdot p = R(p)$ → use this definition because $q(p)$ is given [Ans: 50 units]

$R(p) = 10000pe^{-0.02p}$

$R'(p) = 10000\left(e^{-0.02p} + p(-0.02)e^{-0.02p}\right) = 10000e^{-0.02p}(1-0.02p)$

$R'(p) = 0$ when $1 = 0.02p \Rightarrow p = 50$ #

(Note there are many ways to check if this is a maximum)

Ex.34: A demand function is $p = f(q) = 100 - 0.05q$ unit. A supply function is $p = g(q) = 10 + 0.1q$ bahts/unit. Determine consumers' and producers' surplus under market equilibrium.

[Ans: 600 units, 70 bahts/unit, 9000 bahts, 18000 bahts]

Solution: At equilibrium point (q_0, p_0), the demand and supply functions are intercepted.

$$100 - 0.05q = 10 + 0.1q$$
$$0.15q = 90$$
$$q_0 = 600 \text{ units}$$
$$p_0 = 10 + 0.1q_0 = 10 + 0.1(600)$$
$$p_0 = 70 \text{ bahts/unit}$$

$$CS = \int_0^{q_0}[f(q) - p_0]dq = \int_0^{600}[\{100 - 0.05q\} - 70]dq = \left[30q - \frac{0.05q^2}{2}\right]_0^{600} = \left[30(600) - \frac{0.05(600)^2}{2}\right] - 0 = 9{,}000 \text{ bahts}$$

$$PS = \int_0^{q_0}[p_0 - f(q)]dq = \int_0^{600}[70 - \{10 + 0.1q\}]dq = \left[60q - \frac{0.1q^2}{2}\right]_0^{600} = \left[60(600) - \frac{0.1(600)^2}{2}\right] - 0 = 18{,}000 \text{ bahts}$$

Ex.35: A demand function is $p = \dfrac{50}{q+5}$ unit. A supply function is $p = \dfrac{q}{10} + 4.5$ bahts/unit. Determine consumers' and producers' surplus under market equilibrium.

[Ans: 5 units, 5 bahts/unit, 50ln2-25 bahts, 1.25 bahts]

Ex.36: A demand function is $p = 2^{11-q}$. A supply function is $p = 2^{q+1}$. Determine consumers' surplus under market equilibrium.

[Ans: 5 unit, 64 bahts/unit, $\left(\dfrac{2^{11}}{\ln 2} - \dfrac{64}{\ln 2} - 320\right)$ bahts]