# Macroeconomics

Lecture 1

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# Dynamic Problem

- Max $U_0(x_0, c_0) + U_1(x_1, c_1) + .... + U_T(x_T, c_T) + W_0(x_{T+1})$
- Subject to (i) $x_0$ given,
- and (ii) the transition equations
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$$x_1 = g_0(x_0, c_0),$$

$$x_2 = g_1(x_1, c_1),$$

$$\vdots$$

$$x_T = g_{T-1}(x_{T-1}, c_{T-1}),$$

$$x_{T+1} = g_T(x_T, c_T)$$

# Dynamic Problem

- Assume that:
- (1) there is a time separable function, $U_t(x_t,c_t)$, which is a concave function.

  Let $x_t$ be a state variable at time t, t=0,1,…,T.

  Let $c_t$ be a control variable at time t, t=0,1,…,T.

# Dynamic Problem

- (2) there is a transition function whose variables are defined by the set
- $\{x_{t+1}, x_t, c_t \mid x_{t+1} \leq g_t(x_t, c_t), c_t \in R\}$
- which is convex and compact set. (See definitions of these terms from Takayama, A . "Mathematical Economics" (1985), p.20 and p.32).
- Then

# Dynamic Problem

- The values of $U_s(x_s, c_s)$ for $s \geq t$ are independent to the past values of $c_v$ and $x_v$ for $v < t$. This is the consequence of the time separable structure of (1.1) and (1.2)
- Forming the Lagrangian

$$L = \sum_{t=0}^{T} U_t(x_t, c_t) + W_0(x_{T+1}) + \sum_{t=0}^{T} \lambda_t \left[ g_t(x_t, c_t) - x_{t+1} \right], \quad (1.3)$$

- where $\lambda_t$ is a Lagrange multipliers for t = 0,1,…,T.

# Two Methods

- A system of simultaneous equations,
- Backward Recursive method.

# A system of simultaneous equations

# Simultaneous Equations

$$L = \sum_{t=0}^{T} U_t\left(x_t, c_t\right) + W_0\left(x_{T+1}\right) + \sum_{t=0}^{T} \lambda_t\left[g_t\left(x_t, c_t\right) - x_{t+1}\right], \qquad (1.3)$$

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*Note that, one can set*

$$U_t\left(x_t, c_t\right) = x_t . U\left(c_t\right),$$

*where* $x_t = \left(1+r\right)^{-t} = discount\ \ factor.$

# Simultaneous Equations

- The 1st-order conditions

$$\frac{\partial L}{\partial c_t}=0, \quad \frac{\partial U_t(x_t,c_t)}{\partial c_t}+\frac{\partial g_t(x_t,c_t)}{\partial c_t}\lambda_t=0, \qquad t=0,\dots,T \qquad (1.4a)$$

$$\frac{\partial L}{\partial \lambda_t}=0, \quad x_{t+1}=g_t(x_t,c_t), \qquad t=0,1,\dots,T, \qquad (1.4b)$$

# Simultaneous Equations

- There are $2(T+1)$ unknowns :
  - $c_t$ for $t = 0,1,2,\ldots T,$
  - and $\lambda_t$ for $t = 0,1,2,\ldots,T.)$
- There are $2(T+1)$ equations.
- Unique optimal solution exists for each unknown.

# Simultaneous Equations

- Optimal policies

$$c_0^* = h_0(x_0, x_1, ...., x_T), \qquad \lambda_0^* = k_0(x_0, x_1, ...., x_T),$$

$$c_1^* = h_1(x_0, x_1, ...., x_T), \qquad \lambda_1^* = k_1(x_0, x_1, ...., x_T),$$

$$c_2^* = h_2(x_0, x_1, ...., x_T), \qquad \lambda_2^* = k_2(x_0, x_1, ...., x_T), \qquad (1.4c)$$

$$\vdots$$

$$c_T^* = h_T(x_0, x_1, ...., x_T), \qquad \lambda_T^* = k_T(x_0, x_1, ...., x_T),$$

# Backward recursive method

# Recursive method

- $$\frac{\partial L}{\partial c_t} = \frac{\partial U_t(x_t, c_t)}{\partial c_t} + \frac{\partial g_t(x_t, c_t)}{\partial c_t}\lambda_t = 0,\ t = 0,\ldots,T \qquad (1.4d)$$

$$\frac{\partial L}{\partial x_t} = \frac{\partial U_t(x_t, c_t)}{\partial x_t} + \frac{\partial g_t(x_t, c_t)}{\partial x_t}\lambda_t - \lambda_{t-1} = 0,\ t = 1,\ldots,T$$

- $$\frac{\partial L}{\partial x_{T+1}} = W_0'(x_{T+1}) - \lambda_T = 0, \qquad (1.4e)$$

- $$\frac{\partial L}{\partial \lambda_t} = 0, \quad x_{t+1} = g_t(x_t, c_t), \quad t = 0,1,\ldots,T, \qquad (1.4f)$$

# Recursive method

- From (1.4e), one can have

$$\lambda_t = \frac{\partial U_{t+1}\left(x_{t+1}, c_{t+1}\right)}{\partial x_{t+1}} + \frac{\partial g_{t+1}\left(x_{t+1}, c_{t+1}\right)}{\partial x_{t+1}} \lambda_{t+1}$$

- Using this and (1.4d) recursively to eliminate $\lambda_t$, t=0,1,..,T-1.
- From (1.4d) and (1.4f), one has

$$\frac{\partial U_t\left(x_t, c_t\right)}{\partial c_t} + \frac{\partial g_t\left(x_t, c_t\right)}{\partial c_t}\left\{\frac{\partial U_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}}\left[\frac{\partial U_{t+2}}{\partial x_{t+2}} + \frac{\partial g_{t+2}}{\partial x_{t+2}}\left\{\frac{\partial U_{t+3}}{\partial x_{t+3}} + \frac{\partial g_{t+3}}{\partial x_{t+3}}\left\{\ldots + \frac{\partial g_T}{\partial x_T}\left[W_0^{'}\left(x_{T+1}\right)\right]\right\}\ldots\right\}\right]\right\} = 0,$$

$$for \quad t = 0,1,\ldots,T.$$

(1.5a)

# Recursive method

- $$x_{t+1} = g_t\left(x_t, c_t\right), \qquad t = 0,1,\ldots,T.$$ (1.5b)

- At t =T, we have from Eq. (1.4d), (1.5b), that

- $$\frac{\partial U_T\left(x_T, c_T\right)}{\partial c_T} + \frac{\partial g_T\left(x_T, c_T\right)}{\partial c_T} W_0^{'}\left(x_{T+1}\right) = 0$$ (1.5c)

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- $$x_{T+1} = g_T\left(x_T, c_T\right)$$ (1.5d)

# Recursive method

- Now, consider the following optimization problem at T

$$\max \quad U_T(x_T, c_T) + W_0(x_{T+1}) \qquad (1.6a)$$

$$s.t. \quad x_T \quad given,$$

$$x_{T+1} = g_T(x_T, c_T) \qquad (1.6b)$$

- It should be pointed out that this problem is a sub-problem of problem (1.1) and (1.2)

# Lagrangian function

$$L_T = U_T(x_T, c_T) + W_0(x_{T+1}) + \lambda_T\left[g_T(x_T, c_T) - x_{T+1}\right], \quad given \;\; x_T$$

$$\frac{\partial L_T}{\partial c_T} = \frac{\partial U_T(x_T, c_T)}{\partial c_T} + W_0^{'}(x_{T+1})\frac{\partial g_T(x_T, c_T)}{\partial c_T} = 0,$$

$$x_{T+1} = g_T(x_T, c_T)$$

# Recursive method

- For t =T-1, the optimization problem at T-1 is

$$\max \quad U_{T-1}\left(x_{T-1}, c_{T-1}\right) + U_T\left(x_T, c_T\right), \qquad (1.1aa)$$

$$s.t. \quad x_{T-1} \quad given,$$

$$\left.\begin{aligned} x_T &= g_{T-1}\left(x_{T-1}, c_{T-1}\right), \\ x_{T+1} &= 0 = g_T\left(x_T, c_T\right). \end{aligned}\right\} \qquad (1.2aa)$$

- This problem is, again, a sub-problem of problem (1.1) and (1.2)

# Recursive method

$$\frac{\partial L}{\partial c_{T-1}} = \frac{\partial U_{T-1}\left(x_{T-1}, c_{T-1}\right)}{\partial c_{T-1}} + \frac{\partial g_{T-1}\left(x_{T-1}, c_{T-1}\right)}{\partial c_{T-1}} \lambda_{T-1} = 0, \qquad (1.7aaa)$$

$$\frac{\partial L}{\partial c_{T}} = \frac{\partial U_{T}\left(x_{T}, c_{T}\right)}{\partial c_{T}} + \frac{\partial g_{T}\left(x_{T}, c_{T}\right)}{\partial c_{T}} \lambda_{T}$$

$$\frac{\partial L}{\partial x_{T}} = \frac{\partial U_{T}\left(x_{T}, c_{T}\right)}{\partial x_{T}} + \frac{\partial g_{T}\left(x_{T}, c_{T}\right)}{\partial x_{T}} \lambda_{T} - \lambda_{T-1} = 0, \qquad (1.7bbb)$$

$$\frac{\partial L}{\partial x_{T+1}} = \frac{\partial W_{0}\left(x_{T+1}\right)}{\partial x_{T+1}} - \lambda_{T} = 0 \qquad (1.7ccc)$$

$$x_T = g_{T-1}\left(x_{T-1}, c_{T-1}\right), \qquad (1.7ddd)$$

$$x_{T+1} = g_T\left(x_T, c_T\right)$$

# Recursive method

- By using (1.4aaa),(1.4bbb) and (1.4ccc), we have, at t=T-1, that

$$\frac{\partial U_{T-1}\left(x_{T-1},c_{T-1}\right)}{\partial c_{T-1}}+\frac{\partial g_{T-1}\left(x_{T-1},c_{T-1}\right)}{\partial c_{T-1}}\left[\frac{\partial U_{T}\left(x_{T},c_{T}\right)}{\partial x_{T}}+\frac{\partial g_{T}\left(x_{T},c_{T}\right)}{\partial x_{T}}W_{0}^{'}\left(x_{T+1}\right)\right]=0$$

(1.5ccc)

Equation (1.5ccc) can also be obtained from (1.5a), given that t=T-1. Then recall (1.4ddd)

$$x_{T}=g_{T-1}\left(x_{T-1},c_{T-1}\right). \qquad (1.5ddd)$$

# Recursive method

- Next, use $h_T(x_T)$ and $g_{T-1}(x_{T-1}, h(x_{T-1}))$ to replace $c_T$ and $X_T$ in (1.5ccc), we then have

$$\frac{\partial U_{T-1}\left(x_{T-1},c_{T-1}\right)}{\partial c_{T-1}}+\frac{\partial g_{T-1}\left(x_{T-1},c_{T-1}\right)}{\partial c_{T-1}}\left[\frac{\partial U_T\left(x_T,h_T\left(x_T\right)\right)}{\partial x_T}+\frac{\partial g_T\left(x_T,c_T\right)}{\partial x_T}W_0^{'}\left(x_{T+1}\right)\right]=0, \qquad (1.5ccc*)$$

$$x_T = g_{T-1}\left(x_{T-1},c_{T-1}\right), \qquad (1.5ddd*)$$

# Recursive method

- Given $x_{T-1}$ , equations (1.5ccc*) and equations (1.5ddd*) form a system of 2 equations in ($x_T$, $c_{T-1}$).
- Solving these 2 equations for $x_T$ and $c_{T-1}$ as functions of $x_{T-1}$:

- $x_T = f_{T-1}(x_{T-1})$,
- $c_{T-1} = h_{T-1}(x_{T-1})$