Problem 1. Consumption and Leisure Decision. (58 points) In class, we considered separately a consumption decision between goods x_1 and x_2 and a lesisure decision between consumption good x and leisure l. Now we consider those together. Yingyi likes three goods: consumption goods x_1, x_2, and leisure l. He maximizes the utility function

$$u(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} l^{\gamma},$$

with $0 < \alpha_i < 1$ for $i = 1, 2$ and $0 < \gamma < 1$. The consumption good x_i has price p_i (for $i = 1, 2$), the hourly wage is w and the individual has total income M.

1. Write down the budget constraint. Consider that Yingyi has H hours to work and, if he does not work, he takes leisure. For example, H could be 24 hours if the time period is a day. Hence, the hours worked h equal $H - l$. There are no sources of income other than income from hours worked. Write down the budget constraint as a function of x_1, x_2, and l. [Hint: Money spent on goods has to be smaller than or equal to money earned] (5 points)

2. Write down the maximization problem of the worker with respect to x_1, x_2, and l with *all* the relevant constraints Assume that the budget constraint is satisfied with equality. Why can we assume that the budget constraint is satisfied with equality? Provide as complete an explanation as you can. (5 points)

3. Write down the Lagrangean and derive the first order conditions with respect to x_1, x_2, l, and λ. (4 points)

4. Solve for x_1^* as a function of the prices p_1, p_2, w, the total number of hours H, and the parameters α_1, α_2, and γ. [Hint: combine the first and second first-order condition, then combine the first and third first-order condition, and finally plug in budget constraint] Similarly solve for x_2^* and l^*. (6 points)

5. Plot the Engel function relating the demand for good 1 $x_1^*(H)$ to the number of hours available H. (Plot x_1 in the x axis and H in the y axis) In what sense H plays the role of income? Explain. (5 points)

6. Plot the demand function for good 1 $x_1^*(p_1)$ as a function of p_1. (Put x_1 in the x axis and price p_1 in the y axis) Is the demand function downward sloping? Interpret. (5 points)

7. Are goods x_1 and x_2 gross complements, gross substitutes, or neither? Define and answer. (5 points)

8. Plot the demand function for leisure $l^*(w)$ as a function of its (shadow) price w. (Put l in the x axis and price w in the y axis) Is the demand function downward sloping? Interpret. (6 points)

9. Relate the response to question 8 (leisure l and price w) to substitution and income effects. Be careful here, this is not exactly the case we saw in class. (6 points)

10. Using the envelope theorem, compute how the indirect utility $v(p_1, p_2, w, \alpha_1, \alpha_2, \gamma, H)$ changes as H changes: $\partial v / \partial H$. Remember that the indirect utility is the utility of Yingyi at the optimum level of the parameters: $v(p_1, p_2, w, \alpha_1, \alpha_2, \gamma, H) = u(x_1^*, x_2^*, l^*)$. What is the sign of $\partial v / \partial H$? Interpret (6 points)

11. (Extra credit) Solve for the Lagrangean multiplier $\lambda^*(p_1, p_2, w, \alpha_1, \alpha_2, \gamma)$ using the first order conditions above. Comment on what this implies for $\partial v / \partial H$. (6 points)

Solution to Problem 1.

1. The budget constraint is

$$p_1x_1 + p_2x_2 \leq w\left(H - l\right),$$

which can be rewritten as

$$p_1x_1 + p_2x_2 + wl \leq Hw$$

2. Yingyi maximizes

$$\begin{aligned}
\max_{x_1,x_2,l} u\left(x_1, x_2, x_3\right) &= x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma} \\
s.t. p_1x_1 + p_2x_2 + wl &\leq Hw \\
s.t.\ x_1 &\geq 0 \\
s.t.\ x_2 &\geq 0 \\
s.t.\ l &\geq 0
\end{aligned}$$

3. The Lagrangean is

$$L\left(x_1, x_2, x_3, \lambda\right) = x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma} - \lambda\left(p_1x_1 + p_2x_2 + wl - Hw\right).$$

The first order conditions are

$$\begin{aligned}
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2}l^{\gamma} - \lambda p_1 = 0 \\
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_2x_1^{\alpha_1}x_2^{\alpha_2-1}l^{\gamma} - \lambda p_2 = 0 \\
f.o.c. \text{ with respect to } x_1 &: \quad \gamma x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma-1} - \lambda w = 0 \\
f.o.c. \text{ with respect to } \lambda &: \quad -\left(p_1x_1 + p_2x_2 + wl - Hw\right) = 0
\end{aligned}$$

4. Following the hints provided:

- From the first two f.o.c. we derive

$$\frac{\alpha_1}{\alpha_2}\frac{x_2}{x_1} = \frac{p_1}{p_2} \text{ which implies } x_2 = \frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1.$$

- From the first and third f.o.c. we derive

$$\frac{\alpha_1}{\gamma}\frac{l}{x_1} = \frac{p_1}{w} \text{ which implies } l = \frac{p_1}{w}\frac{\gamma}{\alpha_1}x_1.$$

- Substituting the solutions for x_2 and x_3 into the budget constraint we obtain

$$p_1x_1 + p_2\left(\frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1\right) + w\left(\frac{p_1}{w}\frac{\gamma}{\alpha_1}x_1\right) = Hw$$

which can be simplified to

$$p_1x_1 + p_1\frac{\alpha_2}{\alpha_1}x_1 + p_1\frac{\gamma}{\alpha_1}x_1 = Hw$$

which implies

$$x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \gamma}\frac{Hw}{p_1}.$$

- Using the expressions above for x_2 and x_3, we obtain

$$\begin{aligned}
x_2^* &= \frac{\alpha_2}{\alpha_1 + \alpha_2 + \gamma}\frac{Hw}{p_2} \text{ and} \\
l^* &= \frac{\gamma}{\alpha_1 + \alpha_2 + \gamma}\frac{Hw}{w} = \frac{\gamma}{\alpha_1 + \alpha_2 + \gamma}H.
\end{aligned}$$

5. As we can see,
$$\partial x_1^*/\partial H = \frac{\alpha_1}{\alpha_1+\alpha_2+\gamma}\frac{w}{p_1}$$
which is positive and independent of H. The Engel curve then is linear and increasing. In this case, the number of hours available for work has a relationship to demand akin to that of income because the individual has an implicit income of Hw.

6. As we can see,
$$\partial x_1^*/\partial p_1 = -\frac{\alpha_1 Hw}{(\alpha_1+\alpha_2+\gamma)\, p_1^2}$$
The demand function for x_1^* is downward sloping in price p_1. The higher the price, the less consumers want to purchase of the good that is now more expensive. This tells us that x_1 is not a Giffin good, but we knew this from 5, above, because there we proved that good one is a normal good, and a Giffin good must be inferior.

7. Good one is a gross complement (substitute) for good two if $\partial x_1^*/\partial p_2 > 0$ (< 0). In this case, $\partial x_1^*/\partial p_2 = 0$ (and also $\partial x_2^*/\partial p_1 = 0$), hence the two goods are neither gross substitutes nor gross complements. This is a feature of Cobb-Douglas preferences.

8. The demand function for l^* is independent of w, an unusual feature. This is because there are two opposing effects: (i) a substitution effect that leads to a reduction of leisure time when the shadow cost of leisure w goes up; (ii) an income effect that leads to an increase in leisure when an increase in the wage w occurs. For a Cobb-Douglas function these two effects cancel each other out. This is unusual because in this model a change in the shadow price w increases the income available (Hw), in addition to changing prices. To see this explicitly, consider the following:
   - Applying the envelope theorem to the expenditure minimization problem we get
   $$\frac{\partial e}{\partial w} = \frac{\partial}{\partial w}\left[p_1x_1+p_2x_2+w\,(l-H)-\lambda\left(x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma}-\overline{u}\right)\right] = l^*-H$$
   - Plugging this into the Slutsky equation we get
   $$\frac{\partial l^*}{\partial w} = \frac{\partial h_l^*}{\partial w} - \frac{\partial l^*}{\partial M}\,(l^*-H)$$
   - Note that, unlike what we have seen before, the last term is negative, which means that the income effect is *positive* instead of negative. Increasing the "price" of leisure increases income.

9. See the answer to question 8, above.

10. The envelope theorem states that it is enough to compute the partial derivative of the Lagrangean function with respect to H. Hence,
$$\frac{\partial v^*}{\partial H} = \frac{\partial}{\partial H}\left[x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma}-\lambda\,(p_1x_1+p_2x_2+wl-Hw)\right] = \lambda^* w > 0.$$
More total hours means that Yingyi can increase consumption or leisure or both, making him unambiguously better off, since his preferences are monotonic.

11. To solve for the Lagrangean multiplier λ^*, notice that from the first f.o.c.,
$$\lambda^* = \frac{\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}l^{\gamma}}{p_1}.$$
Substituting the values for x_1^*, x_2^*, and l^*, we get
$$\begin{aligned}\lambda^* &= \frac{\alpha_1\left(\frac{\alpha_1}{\alpha_1+\alpha_2+\gamma}\frac{Hw}{p_1}\right)^{\alpha_1-1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2+\gamma}\frac{Hw}{p_2}\right)^{\alpha_2}\left(\frac{\gamma}{\alpha_1+\alpha_2+\gamma}H\right)^{\gamma}}{p_1} = \\ &= \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\gamma^{\gamma}\left(\frac{1}{\alpha_1+\alpha_2+\gamma}\right)^{\alpha_1+\alpha_2+\gamma-1}(H)^{\alpha_1+\alpha_2+\gamma-1}(w)^{\alpha_1+\alpha_2-1}\left(\frac{1}{p_1}\right)^{\alpha_1}\left(\frac{1}{p_2}\right)^{\alpha_2}\end{aligned}$$

which in turn means that

$$\lambda^* w = \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \gamma^\gamma \left(\frac{1}{\alpha_1 + \alpha_2 + \gamma} \right)^{\alpha_1 + \alpha_2 + \gamma - 1} (H)^{\alpha_1 + \alpha_2 + \gamma - 1} (w)^{\alpha_1 + \alpha_2} \left(\frac{1}{p_1} \right)^{\alpha_1} \left(\frac{1}{p_2} \right)^{\alpha_2}.$$

Notice that the marginal utility of extra time is increasing in the wage, w, and decreasing in the prices of the two goods, p_1, and p_2. This makes sense. When Yingyi has a high wage, he can transform additional time more easiliy into utility from consumption. Conversely, when prices of goods are high, it is harder for Yingyi to transform extra time into utility from consumption.

Problem 2. (28 points) **Complements and Substitutes.**

1. (a) Define when two goods x_1 and x_2 are gross substitutes/complements and net substitutes/complements. (4 points)
   (b) Is it possible for two goods to be gross substitutes and net complements if both goods are normal goods? And if both goods are inferior goods? Use the general form of the Slutsky equation and provide an explanation for the result. (10 points)

2. Kim has utility function $u(x_1, x_2) = \exp(x_1 + x_2)$ for $x_i \geq 0$, $i = 1, 2$.
   (a) Plot the indifference curves of Kim. What kind of goods do they represent? (4 points)
   (b) Are the preferences represented by this utility function monotonic? Define. (4 points)
   (c) Are they rational? Define. (6 points)

Solution to Problem 2.

1. (a) Good one is a gross complement (substitute) for good two if $\partial x_1^*/\partial p_2 > 0$ (< 0) and vice versa. (Note that it is possible for good one to be a gross complement for good two while good two is a gross substitute for good one since one could be normal while the other is inferior.) Meanwhile, goods one and two are net complements (substitutes) if $\partial h_1^*/\partial p_2 > 0$ (< 0) where h_1^* is the Hicksian demand for good one. (Note that it is always true that $\partial h_1^*/\partial p_2 = \partial h_2^*/\partial p_1$ for any two goods, so net complementarity and/or substitutability are always symmetric.)
   (b) First consider the general form of the Slutsky equation for consumption goods:
   $$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial M} x_j^*$$
   If the two goods are gross substitutes we have $\frac{\partial x_i^*}{\partial p_j} > 0$, and if they are net complements we have $\frac{\partial h_i^*}{\partial p_j} < 0$. By the non-negativity of demand we have $x_j^* \geq 0$. If both goods are normal we have $\frac{\partial x_i^*}{\partial M} \geq 0$ so there is no way to get $\frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial M} x_j^* > 0$ as required, so the answer is no, it is not possible. However, if both goods are inferior we have $\frac{\partial x_i^*}{\partial M} \leq 0$ so that if $-\frac{\partial x_i^*}{\partial M} x_j^*$ is big enough we can get $\frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial M} x_j^* > 0$ so the answer is yes, it is possible. (Note that two goods can be net complements in a model with more than two goods.)

2. Kim's preferences:
   (a) The indifference curves are straight lines with slope -1. The goods x_1 and x_2 are perfect substitutes, the individual only cares about the sum of the two goods. To see this, remember that we can apply any strictly increasing transformation to a utility function and we will get a new utility function that represents the same preferences. In particular, we take the natural log of Kim's utility function,
   $$\log(\exp(x_1 + x_2)) = x_1 + x_2$$
   This utility function should be familiar to you as representing perfect substitutes.
   (b) The preferences are monotonic if $x_i \geq y_i$ for all i implies $x \succcurlyeq y$. These preferences are indeed monotonic. If $x_i \geq y_i$ for all i, then $\exp(x_1 + x_2) \geq \exp(y_1 + y_2)$. To confirm this, check that $\frac{\partial u}{\partial x_i} = \exp(x_1 + x_2) > 0$ for $i = 1, 2$.
   (c) Preferences are rational if they are complete (for all x, y in the consumption set, X, either $x \succeq y$, or $y \succeq x$) and transitive (for all x, y, and z in X such that $x \succeq y$, and $y \succeq z$ we have $x \succeq z$). It is not necessary to prove these two properties for Kim's preferences because we already know that if preferences can be represented by a utility function, they must be rational.

Problem 1. Utility maximization. (75 points) In this exercise, we consider a standard utility maximization problem with an unusual (for us) income. The utility function is

$$u(x, y) = \alpha \log(x) + (1 - \alpha) \log(y).$$

This function is well-defined for $x > 0$ and for $y > 0$. From now on, assume $x > 0$ and $y > 0$ unless otherwise stated. The price of good x is p_x and the price of good y is p_y. The unusual part is that consumers' income is given not by a monetary budget M, but by endowments of the goods. To be more precise, the consumer starts off owning $\omega_x > 0$ units of good x and $\omega_y > 0$ units of good y. (These numbers are given, treat them as parameters) The consumer can sell the goods so the initial allocation works as a source of wealth. Given this, the income of the consumer is $p_x\omega_x + p_y\omega_y$.

1. Write down the budget constraint. Notice that the budget constraint is standard other than for the income expression, which is given above (5 points)
2. Just for this point, we pick some specific parameter values. Plot the budget lines assuming $\omega_x = 1$, $\omega_y = 1$. Plot first for the case $p_x = 1$, $p_y = 1$, and then plot the budget line for $p_x = 2$, $p_y = 1$. How does the budget line shift? In a new graph, plot the budget constraint for the same two combinations of prices, but assuming $M = 2$. How does the shift in budget line in the case with endowments ω_x and ω_y compare to the standard case with income M? Provide intuition. (10 points)
3. Let's go back to the general case. The consumer maximizes utility subject to the budget constraint with endowments as in point (1). Write down the maximization problem of the consumer with respect to x and y. Explain *briefly* why the budget constraint is satisfied with equality.(5 points)
4. Write down the Lagrangean function. (5 points)
5. Write down the first order conditions for this problem with respect to x, y, and λ. (5 points)
6. Solve explicitly for x^* and y^* as a function of p_x, p_y, ω_x, and ω_y. (5 points)
7. Use the expression for x^* that you obtained in point 6. Differentiate it with respect to p_x, that is, compute $\partial x^*/\partial p_x$. Is this good a Giffen good? (5 points)
8. We are now interested in the sign of $\partial x^*/\partial p_y$. That is, we would like to know if how the demand for one good depends on the price of the other good. Based on this, are these goods (gross) substitutes or complements? (5 points)
9. Consider now how the demand for x^* varies as the endowments ω_x and ω_y vary. That is, compute $\partial x^*/\partial \omega_x$ and $\partial x^*/\partial \omega_y$. Provide an interpretation of these results. (5 points)
10. So far you derived the optimal demands x^* and y^*. Now, consider the demands net of the endowments, that is $z_x^* = x^* - \omega_x$ and $z_y^* = y^* - \omega_y$. These net demands represent the *net demand* for a good. If the net demand is positive, it means that the consumer buys additional quantities relative to the endowment. If the net demand is negative, it indicates that the consumer sells (at least partially) the endowment. Derive expressions for z_x^* and z_y^*. (5 points)
11. Compute how the net demand for good x, that is, z_x^*, varies as the endowment for x varies, that is, compute $\partial z_x^*/\partial \omega_x$. Compare to your answer in point (9) regarding $\partial x^*/\partial \omega_x$ and interpret the difference. (10 points)
12. Suppose now that you have to solve the same problem as above, that is, same budget constraint as in point (1), but the utility function is $v(x, y) = x^a y^{1-\alpha}$. Can you derive the solution for this problem without redoing all the algebra? Explain. (10 point)

Solution to Problem 1.

1. The budget constraint is
$$p_x x + p_y y \leq p_x \omega_x + p_y \omega_y$$

2. The difference from the standard case is that the budget line pivots around the endowment (ω_x, ω_y). In the standard case, it pivots around the axis intercept of the good that does not change price (in this case, good y).

3. The maximization problem is
$$\begin{aligned} & \max_{x,y} \alpha \log(x) + (1-\alpha)\log(y) \\ \text{s.t.} \quad & p_x x + p_y y \leq p_x \omega_x + p_y \omega_y, \\ & x \geq 0 \text{ and } y \geq 0 \end{aligned}$$

   We can write down the budget constraint with equality because the utility function is strictly increasing both in x and y. (Now follows the long explanation, you were not required to give this.) With this utility function, $\partial u(x,y)/\partial x > 0$, and we can by symmetry show $\partial u(x,y)/\partial y > 0$. Given that utility is strictly increasing in both goods, the consumer will never choose a point on the interior of the budget set, i.e., a point $(\hat{x}, \hat{y})$ such that $p_x \hat{x} + p_y \hat{y} < p_x \omega_x + p_y \omega_y$. The reason is that the consumer could choose a point $(\hat{x}', \hat{y}')$ with $\hat{x}' > \hat{x}$ and $\hat{y}' > \hat{y}$ that still satisfies the budget constraint, i.e., such that $p_x \hat{x}' + p_y \hat{y}' \leq p_x \omega_x + p_y \omega_y$. (just pick $(\hat{x}', \hat{y}')$ sufficiently close to $(\hat{x}, \hat{y})$). But, given the monotonicity of u, the bundle $(\hat{x}', \hat{y}')$ provides a higher utility than the bundle $(\hat{x}, \hat{y})$. Therefore the consumer in the optimum will never choose a bundle $(\hat{x}, \hat{y})$ such that $p_x \hat{x} + p_y \hat{y} < p_x \omega_x + p_y \omega_y$. We can therefore limit ourselves to the points with $p_x x + p_y y = p_x \omega_x + p_y \omega_y$.

4. Lagrangean is $L(x, y, \lambda) = \alpha \log(x) + (1-\alpha)\log(y) - \lambda(p_x x + p_y y - p_x \omega_x - p_y \omega_y)$.

5. First order conditions:
$$\begin{aligned} \frac{\partial L}{\partial x} &= \frac{\alpha}{x} - \lambda^* p_x = 0 \\ \frac{\partial L}{\partial y} &= \frac{(1-\alpha)}{y} - \lambda^* p_y = 0 \\ \frac{\partial L}{\partial \lambda} &= -(p_x x + p_y y - p_x \omega_x - p_y \omega_y) = 0 \end{aligned}$$

6. Using the first two first order conditions, we find
$$\frac{\alpha}{1-\alpha}\frac{y^*}{x^*} = \frac{p_x}{p_y}$$
   or
$$y^* = ((1-\alpha)/\alpha)(p_x/p_y)\,x^*. \tag{1}$$
   We substitute this into the budget constraint to get $p_x x^* + p_y ((1-\alpha)/\alpha)(p_x/p_y)\,x^* = M$ or $p_x x^*(1 + (1-\alpha)/\alpha) = p_x \omega_x + p_y \omega_y$ or
$$x^* = \alpha \frac{p_x \omega_x + p_y \omega_y}{p_x} = \alpha\left[\omega_x + \frac{p_y}{p_x}\omega_y\right]. \tag{2}$$
   We substitute (2) into (1) to get
$$y^* = ((1-\alpha)/\alpha)(p_x/p_y)\,\alpha \frac{p_x \omega_x + p_y \omega_y}{p_x} = (1-\alpha)\frac{p_x \omega_x + p_y \omega_y}{p_y} = (1-\alpha)\left[\omega_y + \frac{p_x}{p_y}\omega_x\right]. \tag{3}$$

7. The derivative of x^* with respect to price p_x is
$$\frac{\partial x^*}{\partial p_x} = -\alpha \frac{\omega_y p_y}{p_x^2} < 0.$$
   This good is a not a Giffen good, since the demand for good x is decreasing in its price.

8. The derivative of x^* with respect to price p_y is
$$\frac{\partial x^*}{\partial p_y} = \alpha \frac{\omega_y}{p_x} > 0.$$
The demand for good x increases in the price of good y. Hence, the two goods are (gross) substitutes.

9. The derivatives of x^* with respect to the endowments are
$$\frac{\partial x^*}{\partial \omega_x} = \alpha > 0 \text{ and } \frac{\partial x^*}{\partial \omega_y} = \alpha \frac{p_y}{p_x} > 0.$$
An increase in the endowments makes the agent richer. Since these preferences imply that the goods are normal good, higher endowments of either good lead to more demand.

10. Using the expressions above for x^* and y^*, we obtain
$$\begin{aligned} z_x^* &= x^* - \omega_x = \alpha \left[\omega_x + \frac{p_y}{p_x}\omega_y\right] - \omega_x = \alpha \frac{p_y}{p_x}\omega_y - (1-\alpha)\,\omega_x \\ z_y^* &= x^* - \omega_y = (1-\alpha)\left[\omega_y + \frac{p_x}{p_y}\omega_x\right] - \omega_y = (1-\alpha)\,\frac{p_x}{p_y}\omega_x - \alpha\omega_y \end{aligned}$$

11. The net demand for good x, z_x^*, decreases as the endowment of good x, ω_x, increases. This is unlike what we showed above, in that the demand x^* goes up in ω_x. What is happening is that as the endowment of x goes up, the individuals are richer, but they do not spend all of the extra income into good x, deciding to spend some of it into good y. Formally, the demand for x^* goes up by α for an increase in ω_x, which is less than one. That is why the net demand is actually decreasing in the endowment of x: being richer in x leads to some increase in consumption of x, some increase in y, leading to a decrease in net demand for x.

12. Notice that the utility function $v(x, y)$, which is a standard Cobb-Douglas, is a monotonic transformation of the utility function $u(x, y)$, with $v(x, y) = \exp(u(x, y))$. Hence, the two utility functions represent the same preferences, and thus the two utility maximization problems are identical. As such, the solution would be identical.

Short problem. (Utility Maximization with Graphical Solution) (40 points)

1. Consider an individual with budget constraint $2x + y = 10$. That is, price of x is $p_x = 2$, price of y is $p_y = 1$, and income M equals 10. Plot the budget constraint. (5 points)
2. Plot indifferent curves for the case $u(x, y) = x^2 + y^2$. (Remember: an indifference curve is defined by $u(u, y) - \bar{u} = 0$). (5 points)
3. Using the plots of the indifference curves and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y^*. (Remember that you are maximizing utility subject to the budget constraint and subject to $x \geq 0$ and $y \geq 0$) (10 points)
4. Why in the above case it would not be a good idea to solve the problem with a Lagrangean? (5 points)
5. Now, using the same budget constraint $2x + y = 10$, plot indifference curves for the case $u(x, y) = 4x + y$.(5 points)
6. Using the plots of the indifference curves and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y^*. (5 points)
7. What preferences are the ones in points (2) and (5)? What cases do they indicate? (5 points)

Solution to Short Problem.

1. The budget constraint is simply given by the equation $y = 10 - 2x$, it is a straight line crossing the y axis at 10 and the x axis at 5.
2. These indifference curves are circles centered in the origin.
3. Graphically, one can see that the point of tangency between the indifference curves (the circles) and the budget lines are not optimal points, in fact they would minimize utility. The solution in this case it to go to one of the corner solutions with either $x = 0$ or $y = 0$. The solution for $x = 0$ would be $(x^*, y^*) = (0, 10)$, yielding utility $10^2 = 100$. The solution for $y = 0$ would be $(x^*, y^*) = (5, 0)$, yielding utility $5^2 = 25$. Hence, the solution is $(x^*, y^*) = (0, 10)$, as you should be able to see graphically.
4. If you were to solve the Lagrangrean, given that the preferences are not convex, you would be finding a minimum, not a maximum, you would find the tangency point highlighted above.
5. The indifference curves are straight lines with slope -4
6. The utility-maximizing solutions will once again be at the corners, this time for $(x^*, y^*) = (5, 0)$, yielding utility 20. At the other corner (0,10), the utility is only 10.
7. The preferences in point (2) and examples of the CES family with $\rho = 2$. The preferences in point (5) are cases of perfect substitutes, in which 4 units of good y are equivalent to one unit of good x.

Problem 2. Profit Maximization with Taxes (96 points) We consider the market for widgets, which is characterized by the aggregate (inverse) demand function $p(X) = a - bX$, where X is the total quantity of widgets demanded in the market. The cost function of each company is $c(y) = cy^{\alpha}$, with $c > 0$.

1. Assume perfect competition (that is, the price p of the widget is given) and set up the profit maximization of each firm. (5 points)
2. Solve for the profit-maximizing level of production $y^*(p)$ (that is, the supply function) using the first-order condition. (5 points)
3. Check the second-order conditions. Under what values of the parameters are they satisfied? Interpret the economic significance of this parameter restriction. (5 points)
4. Now consider the conditions for the market equilibrium. For points 4-7, assume that the parameters are such that the second order conditions are satisfied. Assume that N firms produce and write the equation for the equilibrium price p^* that equates aggregate supply and demand. Do not attempt to solve explicitly for p^*. (5 points)
5. Now introduce taxation. Denote by p the price inclusive of tax that the consumer pays, and by $p - t$ the price net of tax that accrues to the producer. Rewrite the market equilibrium condition. (5 points) [Note: If you get stuck here, you can move on to point 8]
6. Use the implicit function theorem to compute $\partial p^*/\partial t$. (5 points)
7. Show that $0 < \partial p^*/\partial t < 1$. What does it mean economically? (5 points)
8. From now on, consider the case $\alpha = 1$. Assume for now no taxes ($t = 0$). What is the economic interpretation of this special case? (5 points)
9. Characterize mathematically the supply function $y^*(p)$ for an individual company, and plot it. (5 points)
10. Solve for the market equilibrium price p^* and total quantity X^* produced in the market, assuming no taxes. [Note: A figure may help you here] (5 points)
11. Compute the aggregate consumer surplus and the producer surplus. You can help yourself with a plot of market demand and supply. [Note: Do not worry here about the distinction between compensated and uncompensated demand] (5 points)
12. Solve now for the market equilibrium price p^* and total quantity X^* produced in the market assuming a tax t. How much of the tax do the companies pass through in prices (that is, what is $\partial p^*/\partial t$)? (5 points)
13. Compute the consumer surplus and the producer surplus for the case with a tax t. (5 points)
14. Compute the total surplus adding the consumer surplus, the producer surplus, and the revenue raised with the taxes. (5 points)
15. Using what you just found, argue that the deadweight loss from taxation is given by $t^2/2b$. Do you agree or disagree with the following statement? Provide a precise argument: 'Small taxes are not very distortionary, but large taxes can induce very large deadweight loss' (8 points)
16. Consider now the case of monopoly. Keep assuming $\alpha = 1$ and a tax t. [Note: As above, the price p denotes the price that consumers pay inclusive of taxes, and $p - t$ is the price that producers receive] Set-up the maximization problem and solve for the profit-maximizing price p_M^* and quantity X_M^*. (8 points)
17. How much of the tax does the firm pass through to the consumer (that is, what is $\partial p^*/\partial t$)? Compare to the case of perfect competition for $\alpha = 1$. (5 points)

18. Compute the producer surplus and the consumer surplus and compare to the case of perfect competition for $\alpha = 1$. (5 points)

Solution of Problem 2.

1. Each firm maximizes
$$\max_y py - c(y) = py - cy^\alpha$$

2. The first-order condition for y^* is
$$p - \alpha c y^{\alpha-1} = 0,$$
leading to the solution $y^* = (p/\alpha c)^{1/(\alpha-1)}$ (so long as α is different from zero)

3. The second-order condition for y^* is
$$-\alpha(\alpha-1)\, c y^{\alpha-2} < 0.$$
This condition is satisfied for $\alpha > 1$. This condition is the condition of decreasing returns to scale.

4. To solve for the equilibrium price, we must equate supply Y and demand X. First, remember that total production is $Y = Ny^* = N(p/\alpha c)^{1/(\alpha-1)}$. One way to do this is to substitute the solution for Y^* from the production decision into the demand function:
$$p - a + bN\left(\frac{p}{\alpha c}\right)^{1/(\alpha-1)} = 0$$

5. The market equilibrium with taxes is
$$p - a + bN\left(\frac{p-t}{\alpha c}\right)^{1/(\alpha-1)} = 0$$

6. We can compute
$$\begin{aligned}\frac{\partial p^*}{\partial t} &= -\frac{-\frac{bN}{\alpha-1}\left(\frac{1}{\alpha c}\right)^{1/(\alpha-1)}(p-t)^{-\frac{\alpha}{\alpha-1}}}{1+\frac{bN}{\alpha-1}\left(\frac{1}{\alpha c}\right)^{1/(\alpha-1)}(p-t)^{-\frac{\alpha}{\alpha-1}}} = \\ &= \frac{\frac{bN}{\alpha-1}\left(\frac{1}{\alpha c}\right)^{1/(\alpha-1)}(p-t)^{-\frac{\alpha}{\alpha-1}}}{1+\frac{bN}{\alpha-1}\left(\frac{1}{\alpha c}\right)^{1/(\alpha-1)}(p-t)^{-\frac{\alpha}{\alpha-1}}}\end{aligned}$$

7. Both numerator and denominator are positive (remember $\alpha > 1$), hence $\partial p^*/\partial t > 0$. Since the denominator equals the numerator plus a positive number (1), the ratio has to be smaller than 1, hence $\partial p^*/\partial t < 1$. This shows that the incidence of the tax is partially on the consumer and partially on the producer.

8. The case $\alpha = 1$ corresponds to the case of constant returns to scale with no fixed costs.

9. In this case the marginal cost function $c'(y)$ equals a constant c. Further, in order for the company to break even, the following condition has to be satisfied:
$$py - cy \geq 0,$$
which implies that the company produces only if $p \geq c$. Hence, the supply function is
$$y^*(p) = \begin{cases} \infty & \text{if } \; p > c \\ \text{any } y \in [0, \infty) & \text{if } \; p = c \\ 0 & \text{if } \; p < c \end{cases}$$

10. Given that the supply function is horizontal, the price is determined by the marginal cost c: $p^* = c$. It follows that the quantity produced X^* is $c = a - bX^*$, or $X^* = (a-c)/b$.

11. The producer surplus is zero, since each firm produces at marginal cost and makes zero profits: $\pi = p^* y^* - cy^* = cy^* - cy^* = 0$. The consumer surplus is the area below the demand curve and above the price. This area is a triangle with base X^* and height $(a - p^*) = (a-c)$. Hence,
$$CS = \frac{(a-c)}{b}\frac{(a-c)}{2} = \frac{(a-c)^2}{2b}.$$

12. In the case with a tax t, the price that the producer is now willing to accept in order to break even is $c+t$. Hence, the price in equilibrium is $p^* = c+t$. The quantity produced X^* is $c+t = a - bX^*$, or $X^* = (a-c-t)/b$.

13. The producer surplus is still zero. The consumer surplus is now
$$CS = \frac{(a-c-t)}{b}\frac{(a-c-t)}{2} = \frac{(a-c-t)^2}{2b}.$$

14. The revenue raised with the taxes is $t * X^* = t(a-c-t)/b$. Hence, the total surplus is
$$\begin{aligned} TS &= \frac{(a-c-t)^2}{2b} + \frac{(a-c-t)\,t}{b} = \frac{(a-c-t)(a-c-t+2t)}{2b} = \\ &= \frac{(a-c-t)(a-c+t)}{2b} = \frac{(a-c)^2 - t^2}{2b} \end{aligned}$$

15. The deadweight loss of taxation is the difference between the total surplus before taxes are introduced, that is, $(a-c)^2/2b$, and the total surplus with taxes, in the above expression. The difference is $-t^2/2b$. This implies that the deadweight loss grows not linearly with the tax, but with the square power. Hence, while small taxes will induce a small distortion, large taxes can induce a large distortion.

16. With a tax t, a monopolist maximizes
$$\max_X (p(X) - t)X - cX = (a - bX - t)X - cX,$$
with first-order conditions
$$a - 2bX^* - t - c = 0,$$
which implies $X^* = (a-c-t)/2b$. The price is
$$p = a - bX^* = a - b\frac{a-c-t}{2b} = \frac{a+c+t}{2}.$$

17. The firm passes through only 1/2 of the tax ($\partial p^*/\partial t = 1/2$), which is half as much as in the case of perfect competition. This is because the monopolist finds it profit-maximizing to bear half of the burden. If it were to pass it on fully, it would lower demand 'too much'.

18. The producer surplus is larger than in the case of perfect competition (where it was zero):
$$\pi = (p^* - c - t)X^* = \left(\frac{a-t-c}{2}\right)\left(\frac{a-t-c}{2b}\right) = \frac{(a-t-c)^2}{4b}.$$
The consumer surplus is smaller than in perfect competition:
$$CS = \frac{a-t-c}{2b}\frac{a - \left(\frac{a+c+t}{2}\right)}{2} = \frac{a-t-c}{2b}\frac{a-c-t}{4} = \frac{(a-t-c)^2}{8b}.$$
The total surplus [not required] is
$$TS = \frac{(a-t-c)^2}{4b} + \frac{(a-t-c)^2}{8b} = \frac{3(a-t-c)^2}{8b}$$
Notice that this is smaller than in the case of perfect competition.

Problem 1. Production (38 points)**.** Consider a farmer that produces corn using labor. The labor cost in dollar to produce y bushels of corn is $c(y) = Cy^2$, with $C > 0$. There are 100 identical farms which all behave competitively. Corn sells at a price p per bushel.

1. Derive the marginal cost $c'(y)$ and the average cost $c(y)/y$. Plot them. Derive graphically the supply curve. (Have the quantity y on the horizontal axis). (5 points)
2. Write the expression for the supply of corn $y(p)$ for each firm, and derive the expression for the aggregate supply function $Y^S(p)$ (5 points)
3. From now on, suppose that the demand curve of corn is $D(p) = 200 - 50p$ and assume $C = 1$. Derive the equilibrium price p^* and total quantity sold Y^{S*}. (5 points)
4. Derive the profits of each firm. (5 points)
5. Now assume that land is not free, and that the farmers are renting the land from a monopolist (that is, there is only one land owner, and it is impossible to rent land somewhere else). How much will the land-owner charge each of the farms? Explain. (8 points)
6. Taking the rent of the land into account, how much are the profits of each firm? (4 points)
7. In what sense there is a parallel between the presence of rents in the land and the entry of firms in the long-run equilibrium? (6 points)

Solution of Problem 1.

1. $c'(y) = 2Cy$ and $c(y)/y = Cy$. Since marginal cost is always larger than average cost, the supply curve is the marginal cost curve. See graph in appendix.
2. The condition $p = c'(y)$ implies

$$\begin{aligned} p &= 2Cy \text{ or} \\ y^*(p) &= \frac{p}{2C} \end{aligned}$$

The aggregate supply function, summing across all firms, is

$$y^*(p) = \sum_{j=1}^{100} \frac{p}{2C} = \frac{50p}{C}$$

3. The equilibrium is

$$\begin{aligned} y^*(p) &= 50p = D(p) = 200 - 50p \\ p(50+50) &= 200 \\ p^* &= 2 \end{aligned}$$

and

$$\begin{aligned} Y^{S*} &= 100 * y(p) \\ &= 200 - 50 * p^* \\ &= 100 \end{aligned}$$

4. Since $Y^{S*} = 100$, each firm produces 1 unit and earns profit $\pi = p * y^* - c(y^*) = 2 - 1^2 = 1$

5. Since land does not enter the corn production function, the land rental price (r) is a fixed cost the farmer must pay to operate. Fixed costs do not change the farmer's profit-maximizing choice of y, but if profit is negative at that y, the farmer will choose not to produce at all (exit the industry).

   In this case, given the profits calculated above, no farmer will pay more than \$1 for land. Every farmer will pay any price less than or equal to \$1, however, because operating will still be profitable. So the monopolist faces a land demand curve that is perfectly inelastic for $r \leq 1$:

$$D(r) = \begin{cases} 100 & \text{if } r \leq 1, \\ 0 & \text{if } r > 1. \end{cases}$$

   Given this demand, the monopolist maximizes his own profits by setting $r = 1$: the monopolist extracts all the farmers' profit as rent.

6. Each farmer's profit will be zero.

7. Both drive farmers' profits to zero, though in the case of rents the surplus is taken by the land-owner while in the case of perfect competition, it becomes consumer surplus.

$$u(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

with $0 < \alpha_i < 1$ for $i = 1, 2, 3$. The consumption good x_i has price p_i (for $i = 1, 2, 3$) and the individual has total income M.

1. Compute the marginal utility of consumption with respect to good x_1, $\partial u(x_1, x_2, x_3) / \partial x_1$. (2 points)
2. What is the limit of the marginal utility for $x_1 \to 0$ and for $x_1 \to \infty$? Interpret the economic intuition behind this feature of this utility function. (5 points)
3. Write the budget constraint. (3 points)
4. Write the maximization problem of Seung. Seung wants to achieve the highest utility subject to the budget constraint. Write down the boundary constraints for x_1, x_2, x_3, and neglect them for now. (3 points)
5. Assuming that the budget constraint holds with equality, write down the Lagrangean and derive the first order conditions with respect to x_1, x_2, x_3, and λ. (5 points)
6. Solve for x_1^* as a function of the prices p_1, p_2, p_3, the income M, and the parameters α_1, α_2, and α_3. [Hint: combine the first and second first-order condition, then combine the first and third first-order condition, and finally plug in budget constraint] Similarly solve for x_2^* and x_3^*. (6 points)
7. Is this true or false? Show: "Cobb-Douglas preferences have the feature that the share of money spent on each good does not depend on the income, or on prices" (6 points)
8. Are the boundary conditions for x_1, x_2, and x_3 satisfied? (2 points)
9. Is good x_1 a normal good (for all values of M and prices p_i)? Compute and answer. (4 points)
10. Plot the implied demand function for x_1, that is plot x_1 as a function of p_1. (Put p_1 on the y axis and x_1 on the x axis) (4 points)
11. Is good x_1 a Giffen good? Why did you know this already from the answer to question 9? (5 points)
12. Are goods x_1 and x_2 gross complements, gross substitutes, or neither? Define and answer. (5 points)

Solution to Problem 1.

1. The marginal utility of consumption $\partial u(x_1, x_2, x_3) / \partial x_1$ is
$$\partial u(x_1, x_2, x_3) / \partial x_1 = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3}.$$
2. For $x_1 \to 0$ the marginal utility converges to $+\infty$ (keep in mind $\alpha_1 < 1$). This means that for very low consumption of x_1, the agent has un unlimited desire for some consumption of that good. For $x_1 \to \infty$ the marginal utility converges to 0. This means that for very high consumption of x_1, the additional unit of consumption has almost no added utility.
3. The budget constraint is
$$p_1 x_1 + p_2 x_2 + p_3 x_3 \leq M.$$

4. Seung maximizes

$$\begin{aligned}
\max_{x_1,x_2,x_3} u(x_1,x_2,x_3) &= x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \\
s.t. p_1x_1 + p_2x_2 + p_3x_3 &\le M \\
s.t.\ x_1 &\ge 0 \\
s.t.\ x_2 &\ge 0 \\
s.t.\ x_3 &\ge 0
\end{aligned}$$

5. The Lagrangean is

$$L(x_1,x_2,x_3,\lambda) = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} - \lambda(p_1x_1 + p_2x_2 + p_3x_3 - M).$$

The first order conditions are

$$\begin{aligned}
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}x_3^{\alpha_3} - \lambda p_1 = 0 \\
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}x_3^{\alpha_3} - \lambda p_2 = 0 \\
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_3 x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-1} - \lambda p_3 = 0 \\
f.o.c. \text{ with respect to } \lambda &: \quad -(p_1x_1 + p_2x_2 + p_3x_3 - M) = 0
\end{aligned}$$

6. From the first two f.o.c. we derive

$$\frac{\alpha_1}{\alpha_2}\frac{x_2}{x_1} = \frac{p_1}{p_2}.$$

which implies

$$x_2 = \frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1.$$

From the first and third f.o.c. we derive

$$\frac{\alpha_1}{\alpha_3}\frac{x_3}{x_1} = \frac{p_1}{p_3}$$

which implies

$$x_3 = \frac{p_1}{p_3}\frac{\alpha_3}{\alpha_1}x_1.$$

Substituting the solutions for x_2 and x_3 in the budget constraint we obtain

$$p_1x_1 + p_2\left(\frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1\right) + p_3\left(\frac{p_1}{p_3}\frac{\alpha_3}{\alpha_1}x_1\right) = M$$

which implies

$$x_1^* = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}\frac{M}{p_1}.$$

Using the expressions above for x_2 and x_3, we obtain

$$\begin{aligned}
x_2^* &= \frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}\frac{M}{p_2} \text{ and} \\
x_3^* &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}\frac{M}{p_3}.
\end{aligned}$$

7. We can re-write the expressions above for x_1^*, x_2^* and x_3^* as follows:

$$\frac{x_i^*p_i}{M} = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}$$

for $i = 1, 2$, and 3. The left-hand side is the share of money M spent on good i, and the right hand side of the equation shows that this is a constant, it does not depend on prices or income. Hence, the statement is true. This is a peculiar feature of Cobb-Douglas preferences.

8. In all of the expressions above for x_1^*, x_2^* and x_3^*, the conditions $x_i^* \geq 0$ are satisfied, and hence the boundary constraints are satisfied.

9. We can compute
$$\partial x_1^*/\partial M = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} \frac{1}{p_1},$$
which is positive for all levels of the parameters, and hence the good is normal.

10. The demand function is a hyperbola, monotnically decreasing as we expect most demand functions to be.

11. Hence, the good in not a Giffen good. We knew this already from the fact that good x_1 is a normal good, that is, $\partial x_1^*/\partial M > 0$. Given the Slutzky equation, we know that a normal good can never be Giffen. Formally,
$$\frac{\partial x_1^*}{\partial p_1} = \frac{\partial h_1^*}{\partial p_1} - \frac{\partial x_1^*}{\partial M} x_i^*.$$
Since $\partial h_1^*/\partial p_1 < 0$ and $\partial x_1^*/\partial M > 0$ (normal good), we know that $\partial x_1^*/\partial p_1 < 0$.

12. Two goods are raw complements (substitutes) if $\partial x_1^*/\partial p_2 > 0$ (< 0). In this case, $\partial x_1^*/\partial p_2 = 0$ (and also $\partial x_2^*/\partial p_1 = 0$), hence the two goods are neither substitutes nor complements. This is a feature of Cobb-Douglas preferences.

Problem 2. (26 points)

1. Angela has utilty function $u(x_1, x_2) = 2x_1 + 2x_2$.

   (a) Plot the indifference curves of Angela. What kind of goods do they represent? (4 points)

   (b) Using the plot you did, find the utility-maximing solution x_1^*, x_2^* for prices $p_1 = 1$, $p_2 = 2$ and income M. Argue the steps you make. (8 points)

2. Kim has utility function $u(x_1, x_2) = \min(x_1, 2x_2)$

   (a) Plot the indifference curves of Kim. What kind of goods do they represent? (4 points)

   (b) Are the preferences represented by this utility function monotonic? Define. (4 points)

   (c) Are they strictly monotonic? Define. (6 points)

Solution to Problem 2.

1. Angela's preferences:

   (a) The indifference curves are straight lines with slope -1. The goods x_1 and x_2 are perfect substitutes, the individual only cares about the sum of the two goods. Do not be fooled by the 2 in front of the utility function, you can just divide the expression by 2 and get back the usual $u(x_1, x_2) = x_1 + x_2$.

   (b) Since good x_2 is more expensive, the individual will never purchase it, and will spend all the money on good x_1. Hence, the solutions are $x_1^* = M/p_1 = M$ and $x_2^* = 0$.

2. Kim's preferences:

   (a) The indifference curves are straight angles. The goods x_1 and x_2 are perfect complements, as for the case of left and right shoe, Kim only cares about x_1 if she also has enough of x_2.

   (b) The preferences are monotonic if $x_i \geq y_i$ for all i implies $x \succcurlyeq y$. These preferences are indeed monotonic. If $x_i \geq y_i$ for all i, then $\min(x_1, 2x_2) \geq \min(y_1, 2y_2)$.

   (c) The preferences are strongly monotonic if $x_i \geq y_i$ for all i and $x_j > y_j$ for some j implies $x \succ y$. Consider $x = (6, 2)$ and $y = (5, 2)$. Clearly, $x_i \geq y_i$ for all i and $x_j > y_j$ for some j, but $x \succ y$ does not hold, since $u(x) = 4 = u(y)$ and hence $x \sim y$.