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CHAPTER 3

The Logic of Quantified Statement

3.1 Predicates and Quantified Statement I

• **Definition**

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example 3.1.1
Finding Truth Values of a Predicate

Let $P(x)$ be the predicate "$x^2 > x$" with domain the set $\mathbf{R}$ of all real numbers. Write $P(2), P\left(\frac{1}{2}\right)$, and $P\left(-\frac{1}{2}\right)$, and indicate which of these statement are true and which are false.

- **Definition**

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

Example 3.1.2
Finding the Truth Set of a Predicate

Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

a. The domain of n is the set $\mathbf{Z}^+$ of all positive integers.

b. The domain of n is the set $\mathbf{Z}$ of all integers.

The Universal Quantifier: ∀

One sure way to change predicates into statements is to assign specific values to all their variables. For example, if x represents the number 35, the sentence "x is (evenly) divisible by 5" is a true statement since 35=5·7. Another way to obtain statements from predicates is to add **quantifiers.** Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.

The Universal Quantifier: ∀

The symbol ∀ denotes "for all" and is called the **universal quantifier.**

• Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example 3.1.3

Truth and Falsity of Universal Statements

a. Let $D = \{1,2,3,4,5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true.

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x$$

Find a counterexample to show that this statement is false.

The Existential Quantifier: ∃

The symbol ∃ denotes "there exists" and is called the **existential quantifier.**

• Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

Example 3.1.4 Truth and Falsity of Existential Statements

a. Consider the statement

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

Show that this statement is true.

Example 3.1.4 Truth and Falsity of Existential Statements

b. Let $E = \{5,6,7,8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Formal Versus Informal Language

It is important to be able to translate from formal to informal language when trying to make sense of mathematical concepts that are new to you. It is equally important to be able to translate from informal to formal language when thinking out a complicated problem.

Example 3.1.5 Translating from Formal to Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀ or ∃.

a. $\forall x \in \mathbf{R}, x^2 \geq 0.$

b. $\forall x \in \mathbf{R}, x^2 \neq -1.$

c. $\exists m \in \boldsymbol{Z}^{+}$ such that $m^2 = m.$

Another way to restate universal and existential statements informally is to place the quantification at the end of the sentence.

For instance, instead of saying "For any real number x, x^2 is nonnegative," you could say "x^2 is nonnegative for any real number x."

In such a case the quantifier is said to "trail" the rest of the sentence.

Example 3.1.6 Trailing Quantifiers

Rewrite the following statement so that the quantifier trails the rest of the sentence.

a. For any integer n, $2n$ is even.

b. There exists at least one real number x such that $x^2 \leq 0$.

Example 3.1.7 Translating from Informal to Formal Language

Rewrite each of the following statements formally. Use quantifiers and variables.

a. All triangles have three sides.

b. No dogs have wings.

c. Some programs are structured.

Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the **universal conditional statement:**

$\forall x$, if $P(x)$ then $Q(x)$.

Example 3.1.8 Writing Universal Conditional Statements Informally

Rewrite the following statement informally, without quantifiers or variables.

$$\forall x \in \mathbf{R}, \text{ if } x>2 \text{ then } x^2 > 4.$$

Example 3.1.9 Writing Universal Conditional Statements Formally

Rewrite each of the following statements in the form ∀_________, if________then________.

a. If a real number is an integer, then it is a rational number.
b. All bytes have eight bits.
c. No fire trucks are green.

Equivalent Forms of Universal and Existential Statements

Example 3.1.10 Equivalent Forms for Universal Statements

Rewrite the following statement in the two forms "$\forall x$, if ______________ then _____________." and "$\forall$ __________ x, _______________": All squares are rectangles.

Example 3.1.11 Equivalent Forms for Existential Statements

A **prime number** is an integer greater than 1 whose only positive integer factors are itself and 1. Consider the statement "There is an integer that is both prime and even."

Let Prime(n) be "n is prime" and Even(n) be "n is even." Use the notation Prime(n) and Even(n) to rewrite this statement in the following two forms:

a. $\exists n$ such that ______________ $\wedge$ __________________.

b. $\exists$ _________________ n such that __________________.

Implicit Quantification

Consider the statement

If a number is an integer, then it is a rational number.

As shown earlier, this statement is equivalent to a universal statement. However, it does not contain the telltale word *all* or *every* or *any* or *each*. The only clue to indicate its universal quantification comes from the presence of the indefinite article a. This is an example of *implicit* universal quantificaition.

Implicit Quantification

Mathematical writing contains many examples of implicitly quantified statements. Some occur, as in the first example above, through the presence of the word *a* or *an*. Others occur in cases where the general context of a sentence supplies part of its meaning.

Implicit Quantification

For example, in an algebra course in which the letter x is always used to indicate a real number, the predicate

If $x > 2$ then $x^2 > 4$

is interpreted to mean the same as the statement

$\forall$real numbers x, if $x > 2$ then $x^2 > 4$.

Implicit Quantification

Mathematicians often use a double arrow to indicate implicit quantification symbolically.

For instance, they might express the above statement as

$$x > 2 \Longrightarrow x^2 > 4.$$

• Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \rightarrow Q(x)$.
- The notation $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

3.2 Predicates and Quantified Statement II

Negations of Quantified Statements

Theorem 3.2.1 Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D,\ Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$

Thus

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Negations of Quantified Statements

Theorem 3.2.2 Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$

Thus

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

Example 3.2.1
Negating Quantified Statements

Write formal negations for the following statements:

a. ∀primes p, p is odd.

b. ∃ a triangle T such that the sum of the angles of T equals 200°

Example 3.2.2 More Negations

Rewrite the following statement formally. Then write formal and informal negations.

No politicians are honest.

Example 3.2.3 Still More Negations

Write informal negations for the following statements:

a. All computer programs are finite.
b. Some computer hackers are over 40.
c. The number 1,357 is divisible by some integer between 1 and 37.

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Negations of Universal Conditional Statements

By definition of the negation of a *for all* statement,

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \to Q(x)). \qquad 3.2.1$$

But the negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$$\sim(P(x) \to Q(x)) \equiv P(x) \wedge \sim Q(x). \qquad 3.2.2$$

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Written less symbolically, this becomes

Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Example 3.2.4 Negating Universal Conditional Statements

Write a formal negation for statement (a) and an informal negation for statement (b).

a. ∀people, if p is blond then p has blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug.

Variants of Universal Conditional Statements

- **Definition**

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example 3.2.5 Contrapositive, Converse, and Inverse of a Universal Conditional Statement

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

Necessary and Sufficient Conditions, Only If

- **Definition**

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

Example 3.2.6
Necessary and Sufficient Conditions.

Rewrite the following statements as quantified conditional statements. Do not use the word *necessary* or *sufficient.*

a. Squareness is a sufficient condition for rectangularity.

b. Being at least 35 years old is a necessary condition for being President of the United States.

Example 3.2.7 Only If

Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

3.4 Arguments with Quantified Statements

If some property is true of *everything* in a set, then it is true of any particular thing in the set.

Universal Modus Ponens

Universal Modus Ponens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore\ Q(a)$.	$\therefore\ a$ makes $Q(x)$ true.

Example 3.4.1
Recognizing Universal Modus Ponens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

k is a particular integer that is even.

$\therefore$ k^2 is even.

Universal Modus Tollens

Universal Modus Tollens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$, for a particular a.	a does not make $Q(x)$ true.
$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Example 3.4.3 Recognizing the Form of Universal Modus Tollens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.

Zeus is not mortal.

∴Zeus is not human.

Example 3.4.4 Drawing Conclusions Using Universal Modus Tollens

Write the conclusion that can be inferred using universal modus tollens.

All professors are absent-minded.

Tom Hutchins is not absent-minded.

$\therefore$ ______________________________.

Proving Validity of Arguments with Quantified Statements

- **Definition**

To say that an *argument form* is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An *argument* is called **valid** if, and only if, its form is valid.

Converse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$. ← invalid conclusion	$\therefore a$ makes $P(x)$ true. ← invalid conclusion

Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$, for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$. ← invalid conclusion	$\therefore a$ does not make $Q(x)$ true. ← invalid conclusion

Creating Additional Forms of Arguments

Universal Transitivity

Formal Version	*Informal Version*
$\forall x\, P(x) \to Q(x)$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x\, Q(x) \to R(x)$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
$\therefore \forall x\, P(x) \to R(x)$.	$\therefore$ Any x that makes $P(x)$ true makes $R(x)$ true.