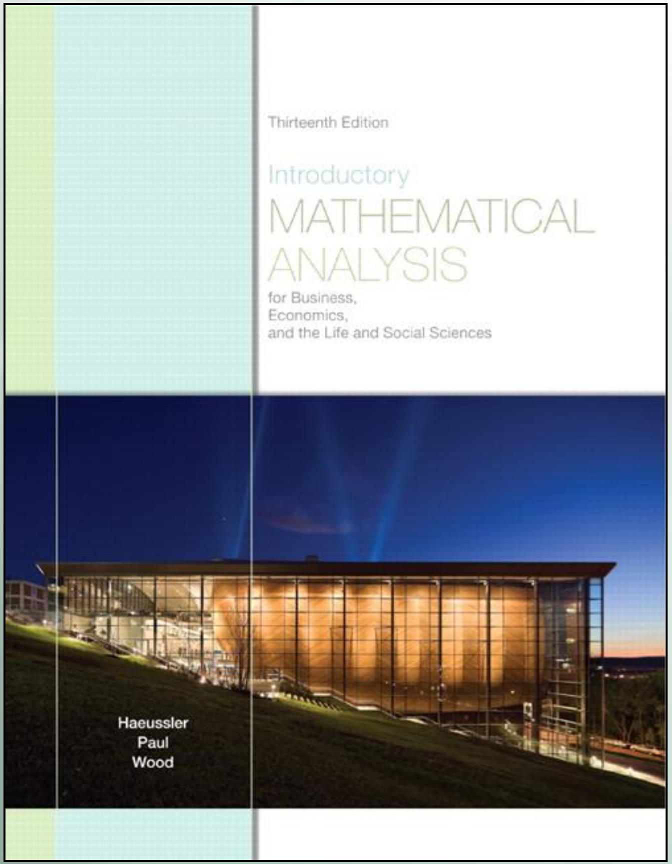
INTRODUCTORY MATHEMATICAL ANALYSIS

For Business, Economics, and the Life and Social Sciences

Chapter 11
Differentiation



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Chapter Objectives

- To compute derivatives by using the limit definition.
- To develop basic differentiation rules.
- To interpret the derivative as an instantaneous rate of change.
- To apply the product and quotient rules.
- To apply the chain rule.

Chapter Outline

11.1) The Derivative

11.2) Rules for Differentiation

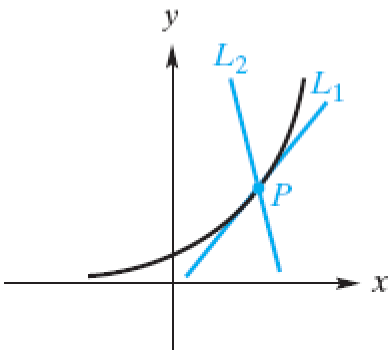
11.3) The Derivative as a Rate of Change

11.4) The Product Rule and the Quotient Rule

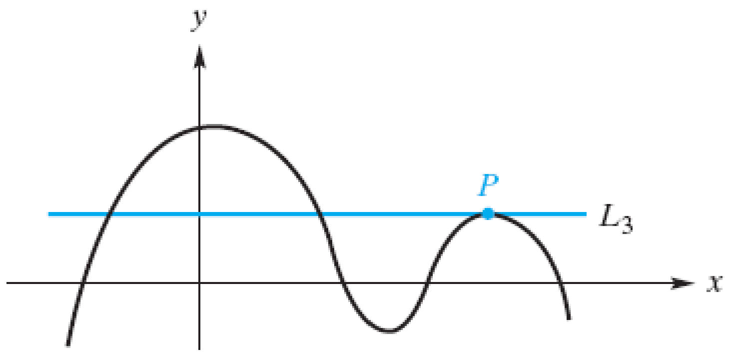
11.5) The Chain Rule and the Power Rule

11.1 The Derivative

- Tangent line at a point:



L_1 is a tangent line at P, but L_2 is not.



L_3 is a tangent line at P.

- The ***slope of a curve*** at *P* is the slope of the tangent line at *P*.
- The slope of the tangent line at (a, $f(a)$) is

$$m_{\tan} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 1 – Finding the Slope of a Tangent Line

Find the slope of the tangent line to the curve $y = f(x) = x^2$ *at the point* (1, 1).

Solution:

- The ***derivative*** of a function *f* is the function denoted *f'* and defined by

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 3 – Finding an Equation of a Tangent Line

If $f(x) = 2x^2 + 2x + 3$, *find an equation of the tangent line to the graph of f at* (1, 7).

Solution:

Example 5 – A Function with a Vertical Tangent Line

Find $\frac{d}{dx}\left(\sqrt{x}\right)$.

Solution:

Example 7 – Continuity and Differentiability

a. For $f(x) = x^2$, it must be continuous for all x.

b. For $f(p) =(1/2)p$, it is not continuous at $p = 0$, thus the derivative does not exist at $p = 0$.

11.2 Rules for Differentiation

- Rules for Differentiation:

RULE 1 Derivative of a Constant:

$$\frac{d}{dx}(c) = 0$$

RULE 2 Derivative of *x*ⁿ:

$$\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$$

RULE 3 Constant Factor Rule:

$$\frac{d}{dx}\left(cf(x)\right) = cf'(x)$$

RULE 4 Sum or Difference Rule

$$\frac{d}{dx} = \left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$$

Example 1 – Derivatives of Constant Functions

a. $\frac{d}{dx}(3) = 0$

b. If $g(x) = \sqrt{5}$, then $g'(x) = 0$.

c. If $s(t) = (1{,}938{,}623)^{807.4}$, then $\frac{ds}{dt} = 0$.

Example 3 – Rewriting Functions in the Form x^n

Differentiate the following functions:

Solution:

a. $y = \sqrt{x}$

b. $h(x) = \dfrac{1}{x\sqrt{x}}$

Example 5 – Differentiating Sums and Differences of Functions

Differentiate the following functions:

a. $F(x) = 3x^5 + \sqrt{x}$

b. $f(z) = \frac{z^4}{4} - \frac{5}{z^{1/3}}$

Example 5 – Differentiating Sums and Differences of Functions

c. $y = 6x^3 - 2x^2 + 7x - 8$

Example 7 – Finding an Equation of a Tangent Line

Find an equation of the tangent line to the curve $y = \dfrac{3x^2 - 2}{x}$ *when x = 1.*

Solution:

11.3 The Derivative as a Rate of Change

- Average velocity is given by $v_{ave} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t}$
- Velocity at time *t* is given by $v = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}$

Example 1 – Finding Average Velocity and Velocity

Suppose the position function of an object moving along a number line is given by $s = f(t) = 3t^2 + 5$, *where t is in seconds and s is in meters.*

a. *Find the average velocity over the interval [10, 10.1].*

b. *Find the velocity when t = 10.*

Solution:

a. When t = 10,

$$v_{ave} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t)-f(t)}{\Delta t}$$

$$= \frac{f(10+0.1)-f(10)}{0.1} = \frac{f(10.1)-f(10)}{0.1} = \frac{311.03 = 305}{0.1} = 60.3 \text{ m/s}$$

b. Velocity at time t is given by

$$v = \frac{ds}{dt} = 6t$$

When t = 10, the velocity is

$$\left.\frac{ds}{dt}\right|_{t=10} = 6(10) = 60\text{m/s}$$

- If $y = f(x)$, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \begin{cases} \text{average rate of change of y} \\ \text{with respect to x over the} \\ \text{interval from } x \text{ to } x+\Delta x \end{cases}$

  then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \begin{cases} \text{instantaneous rate of change of} \\ \text{y with respect to x} \end{cases}$$

Example 3 – Finding a Rate of Change

Find the rate of change of $y = x^4$ with respect to x, and evaluate it when $x = 2$ and when $x = -1$.

Solution:

Example 5 – Rate of Change of Volume

A spherical balloon is being filled with air. Find the rate of change of the volume of air in the balloon with respect to its radius. Evaluate this rate of change when the radius is 2 ft.

Solution:

Applications of Rate of Change to Economics

- **Total-cost** function is c = *f(q)*.
- **Marginal cost** is defined as $\frac{dc}{dq}$.
- **Total-revenue function** is *r* = *f(q)*.
- **Marginal revenue** is defined as $\frac{dr}{dq}$.

Relative and Percentage Rates of Change

- The relative rate of change of *f(x)* is $\frac{f'(x)}{f(x)}$.
- The percentage rate of change of *f (x)* is

$$\frac{f'(x)}{f(x)}(100\%)$$

Example 7 – Marginal Cost

If a manufacturer's average-cost equation is

$$\bar{c} = 0.0001q^2 - 0.02q + 5 + \frac{5000}{q}$$

find the marginal-cost function. What is the marginal cost when 50 units are produced?

Solution:

Example 9 – Relative and Percentage Rates of Change

Determine the relative and percentage rates of change of

$$y = f(x) = 3x^2 - 5x + 25$$

when x = 5.

Solution:

11.4 The Product Rule and the Quotient Rule

The Product Rule

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Example 1 – Applying the Product Rule

Example 3 – Differentiating a Product of Three Factors

The Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Example 5 – Applying the Quotient Rule

If $F(x) = \frac{4x^2+3}{2x-1}$ *, find F'(x).*

Solution:

Example 7 – Differentiating Quotients without Using the Quotient Rule

Differentiate the following functions.

Consumption Function

$$\text{Marginal propensity to consume} = \frac{dC}{dI}$$

$$\text{Marginal propensity to save} = 1 - \text{Marginal propensity to consume}$$

Example 9 – Finding Marginal Propensities to Consume and to Save

If the consumption function is given by $C = \frac{5\left(2\sqrt{I^3}+3\right)}{I+10}$ *determine the marginal propensity to consume and the marginal propensity to save when* $I = 100$.

Solution:

11.5 The Chain Rule

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Power Rule: $\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$

Example 1 – Using the Chain Rule

a. *If* $y = 2u^2 - 3u - 2$ *and* $u = x^2 + 4$, *find* dy/dx.

Solution:

Example 1 – Using the Chain Rule

b. If $y = \sqrt{w}$ and $w = 7 - t^3$, *find* dy/dt.

Solution:

Example 3 – Using the Power Rule

If $y = (x^3 - 1)^7$, *find* y'.

Solution:

Example 5 – Using the Power Rule

If $y = \dfrac{1}{x^2 - 2}$, *find dy/dx.*

Solution:

Example 7 – Differentiating a Product of Powers

If $y = (x^2 - 4)^5 (3x + 5)^4$, *find y'.*

Solution: