Chapter 5 Multivariate Calculus: Applications

In this chapter we will examine three groups of examples:

1) Effectiveness of Monetary Policy with Balanced Budget in Closed Economy:
   a. Fixed Price without Money Market
   b. Fixed Price with Money Market (IS-LM)
2) Monetary Policy Effectiveness
   a. IS-LM Model
3) Tax Incidence in Competitive Market

5.1 Balanced-Budget Multipliers in Closed Economy.

Three cases are discussed to demonstrate the applications of Cramer's Rule and Implicit Function Theorem:
a) a fixed price, closed economy with no money market
b) a fixed price, IS-LM model with money market

- The Balanced-Budget multipliers changes as the model becomes more general
- See end-of-chapter problems analyzing effects of other parameter on endogeneous variables

5.1.1 Simple Keynesian Model: Fixed price, closed economy without money market $C(Y_d) = a + bY_d$

$$Y = C(Y_d) + I + G$$
$$Y_d = Y - T$$
$$G = T$$

solve for Y^*, endog.
exog. G, T, I
C is endog. – once we solve for Y^* we can solve for C^*

Take total differential, with $dI = 0$ and $dG = dT$ dG

$$dY = C'(Y-T)dY - C'(Y-T)dT + \cancel{dI} + dG$$

$=0$

$$(1 - C'(Y-T))dY = (1 - C'(Y-T))dG$$

$$\frac{dY}{dG} = 1.$$

endog.

exog.

Note: We can apply the Implicit Function Theorem to get the same result. That is, we write the implicit function

equilibrium condition
AD
$AS \Rightarrow Y = C(Y-T) + I + G$

$$f(Y(G), G) = Y - C(Y - T) - I - G$$
$$= Y - C(Y - G) - I - G = 0$$

endog. as a function of G

by noting that $G = T$. Let Y^* be the equilibrium income at the level of government expenditure G_0 and tax T_0, i.e., $Y^* =$

$$f(Y^*, G_0) = Y^* - C(Y^* - G_0) - I - G_0 = 0$$

satisfy the eq. condition

we must have $\frac{\partial f}{\partial Y} \neq 0 = 1 - C'(Y^* - G_0) \neq 0$

$Y(G_0)$. By direct application of the Implicit Function Theorem, we have

$\frac{\partial f(Y^*, G_0)}{\partial G} = C'(Y^* - G_0) - 1$

endog. exog.

$$\frac{\partial Y^*}{\partial G} = -\left(\frac{\partial f(Y^*, G_0)}{\partial Y}\right)^{-1} \frac{\partial f(Y^*, G_0)}{\partial G}$$
$$= -(1 - C')^{-1}(C' - 1) = 1.$$

HW Baldani, p. 173, # 6.2.

5.1.2 IS-LM Model In the previous model of 5.1.1, there is no crowding out effect as the investment is exogeneous. The model now is,

endog (Y, r)

exog.

$$\left.\begin{aligned} Y &= C(Y_d) + I(r) + G \quad \text{— real} \\ M &= L(Y, r) \quad \text{— money.} \end{aligned}\right\}$$

where investment is a function of interest rate r with $I'(r) < 0$, M is the money supply, and $L(Y, r)$ is the money demand with $L_Y(Y, r) > 0$ and $L_r(Y, r) < 0$.

Take total differential of the two equations

$$dY = C'dy - C'dT + I'dr + dG$$
$$dM = L_Y dY + L_r dr$$

With balanced budget, $dG = dT$, and the money supply being exogeneous $dM = 0$, we have

no change in Money supply

$$dY = C'dY + (1 - C')dG + I'dr$$
$$0 = L_Y dY + L_r dr,$$

and written in matrix form

$$\begin{bmatrix} 1 - C' & -I' \\ L_Y & L_r \end{bmatrix} \begin{bmatrix} dY \\ dr \end{bmatrix} = \begin{bmatrix} (1 - C')dG \\ 0 \end{bmatrix}.$$

$A = [a_1 \cdots a_n]$

$\boxed{Ax = b}$ $n \times n$

$[a_1 \cdots a_n]\, x = b$

$x_1 = \frac{|b\ a_2 \cdots a_n|}{|A|}$

By Cramer's Rule,

$$dY = \frac{\begin{vmatrix} (1 - C')dG & -I' \\ 0 & L_r \end{vmatrix}}{\begin{vmatrix} 1 - C' & -I' \\ L_Y & L_r \end{vmatrix}} = \frac{(1 - C')L_r}{(1 - C')L_r + L_Y I'} dG,$$

$x_2 = \frac{|a_1\ b\ a_3 \cdots a_n|}{|A|}.$

$\vdots$

and the Balanced-Budget Multiplier is

$$\frac{\partial Y}{\partial G} = \frac{(1 - C')L_r}{(1 - C')L_r + L_Y I'} = \frac{1}{1 + \phi},$$

where $\phi = \frac{L_Y I'}{(1 - C')L_r} > 0$. Thus $0 < \frac{\partial Y}{\partial G} < 1$..

Economic Explanation:

$\Delta G = \Delta T \Longrightarrow$ ΔY as in the simple model
$\Longrightarrow$ Shift in IS curve
$\Longrightarrow$ higher equilibrium r
$\Longrightarrow$ a reduction in I—crowding out effect.

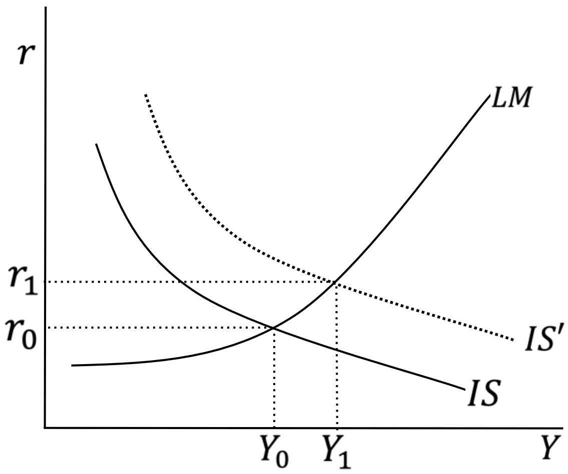

Figure 5.1 Effects of Balance-Budget Multiplier

Note: $\phi = \frac{L_Y I'}{(1-C')L_r} > 0$ depends on four factors: C', I', L_Y and L_r.

HW Repeat the above analysis by applying the Implicit Function Theorem.

HW Baldani, p. 173, # 6.3, 6.4.

5.2 Monetary Policy Effectiveness

This application also illustrates the applications of the Implicit Function Theorem and Implicit Function Differentiation.

Starts with IS-LM Model in a closed economy and find comparative static effects of change in Money Supply: $\frac{\partial Y^*}{\partial M}$ and $\frac{\partial r^*}{\partial M}$.

Then we consider Open Economy with Balance-of-payment constraint (Mundell-Fleming Model), with Fixed and Flexible exchange rate cases under the assumption of perfect capital mobility and perfect capital immobility.

5.2.1 IS-LM Model Equilibrium in goods and money markets of a closed economy.

$$\boxed{\begin{aligned} Y &\cancel{=} C(Y) + I(Y,r) + G_0 \quad = 0 \\ M_0 &\cancel{=} L(Y,r), \quad = 0 \end{aligned}}$$

equilibrium conditions.

C, I, L are also endo's but they are functions of (Y, r). We can find C, I, L later.

Endog. (Y, r) — must be equal to no. of imp. fn's

Exog. (M, G) — must be ≥ 1 variable where

$0 < C' < 1, I_Y > 0, I_r < 0$, and $C' + I_Y < 1$
$L_Y > 0$ and $L_r < 0$.

exog. exog.

Writing the equilibrium income and interest rate as a function of the money supply as $Y^* = Y(M_0)$ and $r^* = r(M_0)$, we have a system of two equations

in equilibrium.

$$Y(M_0) = C\big(Y(M_0)\big) + I\big(Y(M_0), r(M_0)\big) + G_0$$
$$M_0 = L\big(Y(M_0), r(M_0)\big).$$

Take implicit differentiation and suppressing arguments of functions with Y and r being endogeneous and M exogeneous,

$\frac{\partial Y^*}{\partial M} = \frac{\partial Y(M_0)}{\partial M}.$

$$\frac{\partial Y^*}{\partial M} = C'\frac{\partial Y^*}{\partial M} + I_Y\frac{\partial Y^*}{\partial M} + I_r\frac{\partial r^*}{\partial M}$$
$$1 = L_Y\frac{\partial Y^*}{\partial M} + L_r\frac{\partial r^*}{\partial M}.$$

In matrix form,

A x = b

$$\underbrace{\begin{bmatrix} 1 - C' - I_Y & -I_r \\ L_Y & L_r \end{bmatrix}}_{2\times2}\underbrace{\begin{bmatrix} \frac{\partial Y^*}{\partial M} \\ \frac{\partial r^*}{\partial M} \end{bmatrix}}_{2x} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{2\times1}$$

$= x_1$

By Cramer's Rule,

form. — direction of relationship between Y & M.

$$\frac{\partial Y^*}{\partial M} = \frac{\begin{vmatrix} 0 & -I_r \\ 1 & L_r \end{vmatrix}}{\begin{vmatrix} 1 - C' - I_Y & -I_r \\ L_Y & L_r \end{vmatrix}} = \frac{I_r}{L_r(1 - C' - I_Y) + L_Y I_r} > 0$$

$\frac{\partial r^*}{\partial M} = ?$

by noting that $L_r(1 - C' - I_Y) + L_Y I_r < 0$.

<u>Implicit Function Theorem Application</u>: Write the implicit functions

(Y, r) endog.
(M, G) exog.

$\nabla_{\begin{bmatrix} Y \\ r \end{bmatrix}} f(\ ) = \begin{bmatrix} (1 - C' - I_Y) & -I_r \\ -L_Y & -L_r \end{bmatrix}$

$$\mathbf{f}(Y(G, M), r(G, M); G, M) = \begin{bmatrix} Y - C(Y) - I(Y, r) - G \\ M - L(Y, r) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

;
By the Implicit Function Theorem, with equilibrium income $Y^* = Y(G_0, M_0)$ and $r^* = r(G_0, M_0)$ at some given levels of G_0

$f \in C^1$

$\nabla_{\begin{bmatrix} Y \\ r \end{bmatrix}} f(Y^*, r^*; G_0, M_0)$ – nonsingular.

partial derivative of each endog w.r.t each exog.

and M_0 and assuming $\nabla_{\begin{bmatrix}Y\\r\end{bmatrix}}\mathbf{f}(Y^*, r^*; G_0, M_0)$ is invertible, we still satisfy the eq conditions

have function $\begin{bmatrix}Y\\r\end{bmatrix} = \begin{bmatrix}Y(G,M)\\r(G,M)\end{bmatrix}$ such that $Y(G,M), r(G,M)$

a) $\mathbf{f}(Y(G,M), r(G,M); G, M) = \mathbf{0}$, $|G - G_0| < \varepsilon$, and $|M - M_0| < \varepsilon$ for some $\varepsilon > 0$.

b) $\begin{bmatrix}Y^*\\r^*\end{bmatrix} = \begin{bmatrix}Y(G_0,M_0)\\r(G_0,M_0)\end{bmatrix}$, and

endog. exog.

c) $\nabla_{\begin{bmatrix}G\\M\end{bmatrix}}\begin{bmatrix}Y^*\\r^*\end{bmatrix} = -\left[\nabla_{\begin{bmatrix}Y\\r\end{bmatrix}}\mathbf{f}(Y^*, r^*; G_0, M_0)\right]^{-1}\nabla_{\begin{bmatrix}G\\M\end{bmatrix}}\mathbf{f}(Y^*, r^*; G_0, M_0)$

$$\begin{bmatrix}\frac{\partial Y^*}{\partial G} & \frac{\partial Y^*}{\partial M}\\ \frac{\partial r^*}{\partial G} & \frac{\partial Y^*}{\partial M}\end{bmatrix} \quad 2\times 2$$

$$= -\begin{bmatrix}1 - C' - I_Y & -I_r\\ -L_Y & -L_r\end{bmatrix}^{-1}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}_{2\times 2}$$

$$= \frac{1}{L_r(1 - C' - I_Y) + L_Y I_r}\begin{bmatrix}-L_r & I_r\\ L_Y & 1 - C' - I_Y\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \frac{1}{L_r(1 - C' - I_Y) + L_Y I_r}\begin{bmatrix}L_r & I_r\\ -L_Y & 1 - C' - I_Y\end{bmatrix}.$$

We can also solve this by Cramer's Rule.

So, we have as

$$\frac{\partial Y^*}{\partial M} = \frac{I_r}{L_r(1 - C' - I_Y) + L_Y I_r} > 0$$

$$\frac{\partial r^*}{\partial M} = \frac{1 - C' - I_Y}{L_r(1 - C' - I_Y) + L_Y I_r} < 0.$$

$$\frac{\partial Y}{\partial M} = -\frac{\begin{vmatrix}0 & -I_r\\ 1 & -L_r\end{vmatrix}}{\begin{vmatrix}1 - C' - I_Y & -I_r\\ -L_Y & -L_r\end{vmatrix}}$$

Note that we also have

$$\frac{\partial Y^*}{\partial G} = \frac{-L_r}{L_r(1 - C' - I_Y) + L_Y I_r} > 0$$

$$\frac{\partial r^*}{\partial G} = \frac{L_Y}{L_r(1 - C' - I_Y) + L_Y I_r} < 0.$$

Comparative Static Analysis:

- if I_Y increases

$$I_Y \uparrow \Longrightarrow \left(\frac{\partial Y^*}{\partial M} = \frac{I_r}{L_r(1 - C' - I_Y) + L_Y I_r}\right)\uparrow$$

- if I_r decreases, (a bigger negative number)

$$I_r \downarrow \Longrightarrow \left(\begin{aligned}\frac{\partial Y^*}{\partial M} &= \frac{I_r}{L_r(1 - C' - I_Y) + L_Y I_r}\\ &= \frac{1}{\frac{L_r(1 - C' - I_Y)}{I_r} + L_Y}\end{aligned}\right)\uparrow$$

$$\underbrace{\nabla_{\begin{bmatrix} G \\ M \end{bmatrix}} \begin{bmatrix} y^* \\ r^* \end{bmatrix}}_{2\times 2} = -\underbrace{\begin{bmatrix} & \\ & \end{bmatrix}^{-1}}_{2\times 2} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2}$$

$$\begin{bmatrix} \mathbb{x}_1 & \mathbb{x}_2 \end{bmatrix} = A^{-1} \begin{bmatrix} \mathbb{b}_1 & \mathbb{b}_2 \end{bmatrix}$$

$$\mathbb{x}_1 = A^{-1} \mathbb{b}_1 \rightarrow \quad x_1^1 = \frac{|\mathbb{b}_1 \ a_2|}{|A|}, \quad x_2^1 = \frac{|a_1 \ \mathbb{b}_1|}{|A|}.$$

$$\mathbb{x}_2 = A^{-1} \mathbb{b}_2$$

$$\mathbb{x}_1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$$

HW Baldani, p. 174, #6.14, 6.15.

5.3 Tax Incidence in Supply-Demand Model

The standard competitive Supply-Demand model gives 3 important results of tax incidence analysis

1. after tax equilibrium is the same for tax paid by buyers or sellers.
2. consumers and producers shares the burden
3. consumers' burden increases as the elasticity of demand decreases and/or elasticity of supply increases.

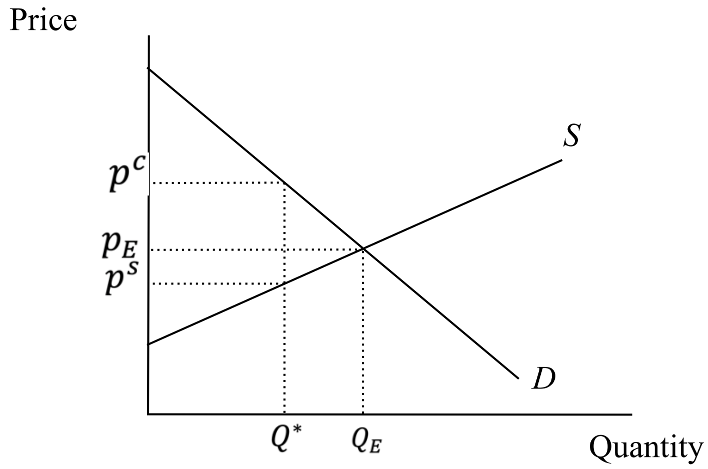

Figure 5.3 Supply-Demand model with specific tax.

p^c = price paid by consumers after tax
p^s = price received by sellers after tax
t = specific tax rate

There are three equations:

$$p^c = D(Q)$$
$$p^s = S(Q)$$
$$p^c = p^s + t,$$

$p^c - p^s = t, \; t > 0.$

with Q, p^c, p^s being endogeneous and t exogeneous and we can write the implicit function

$$\mathbf{f}(Q, p^c, p^s; t) = \begin{bmatrix} p^c - D(Q) \\ p^s - S(Q) \\ p^c - p^s - t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3 implicit functions.
3 endog. Q, P^c, P^s

$$\begin{bmatrix} Q'(t_0) \\ p^{c\prime}(t_0) \\ p^{s\prime}(t_0) \end{bmatrix} = -\left[\nabla_{\begin{bmatrix} Q \\ p^c \\ p^s \end{bmatrix}} f \right]^{-1}$$

① $f \in C^1$

② $\nabla_{\begin{bmatrix} Q \\ p^c \\ p^s \end{bmatrix}} f$ - invertible

For a given tax t_0, let the equilibrium be at (Q^*, p^{c*}, p^{s*}). By Implicit Function Theorem, if the determinant

slope of D ≠ slope of S.

$$\left|\nabla_{\begin{bmatrix} Q \\ p^c \\ p^s \end{bmatrix}} \mathbf{f}(Q, p^c, p^s; t)\right| = \boxed{\begin{vmatrix} -D' & 1 & 0 \\ -S' & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = (D' - S') \neq 0,$$

$D' \neq S'$ $\quad$ $\left.\begin{matrix} D' < 0 \\ S' > 0 \end{matrix}\right\}$ usually

there exist functions $Q(t), p^c(t), p^s(t)$ for $\boxed{|t - t_0|} < \varepsilon$ for some $\varepsilon > 0$ such that, $\quad$ t-exogeneous.

a) $\mathbf{f}(Q(t), p^c(t), p^s(t); t) = \begin{bmatrix} p^c(t) - D(Q(t)) \\ p^s(t) - S(Q(t)) \\ p^c(t) - p^s(t) - t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓ still in equilibrium.

b) $\begin{bmatrix} Q(t_0) \\ p^s(t_0) \\ p^c(t_0) \end{bmatrix} = \begin{bmatrix} Q^* \\ p^{c*} \\ p^{s*} \end{bmatrix}$

c) $\begin{bmatrix} Q'(t_0) \\ p^{s\prime}(t_0) \\ p^{c\prime}(t_0) \end{bmatrix} = \nabla_{\begin{bmatrix} Q \\ p^c \\ p^s \end{bmatrix}} \mathbf{f}(Q, p^c, p^s; t)^{-1} \nabla_{[t]} \mathbf{f}(Q, p^c, p^s; t)$

$$= \begin{bmatrix} -D' & 1 & 0 \\ -S' & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

By Cramer's Rule, if $D' < 0$ and $S' > 0$,

$$\boxed{Q'(t_0)} = -\frac{\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} -D' & 1 & 0 \\ -S' & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{1}{D' - S'} < 0,$$

$$\boxed{p^{c\prime}(t_0)} = -\frac{\begin{vmatrix} -D' & 0 & 0 \\ -S' & 0 & 1 \\ 0 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} -D' & 1 & 0 \\ -S' & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{D'}{D' - S'} > 0,$$

$$p^{s\prime}(t_0) = -\frac{\begin{vmatrix} -D' & 1 & 0 \\ -S' & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} -D' & 1 & 0 \\ -S' & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{S'}{D'-S'} < 0,$$

- If $D' < 0$ and $S' > 0$, then we obtain Result 2 above that the consumers and sellers share the tax burden:

$$0 < p^{c\prime}(t_0) = \frac{D'}{D'-S'} = \frac{1}{1 - S'/D'} < 1$$

$$-1 < p^{s\prime}(t_0) = \frac{S'}{D'-S'} = \frac{1}{D'/S'-1} < 0$$

- We always have $D' < 0$ for market demand and thus if

1. $S' = 0 \Longrightarrow p^{c\prime}(t_0) = 1$ and $p^{s\prime}(t_0) = 0$--the burden is only on the consumers.

2. $S' < 0$ and if

a) $D' < S' < 0$ then $p^{c\prime}(t_0) > 1$ so consumers' burden $> t$, and $p^{s\prime}(t_0) > 0$, meaning producers get higher price.

b) $S' < D' < 0$ then $p^{s\prime}(t_0) < -1$ so producers receive a reduction in price bigger than t, and $p^{c\prime}(t_0) < 0$ so consumers pay less after tax.

Result 3: Burden are shared and depends on elasticities. At the original equilibrium before tax,

$$\eta_D = -\frac{1}{D'}\frac{p_E}{Q_E} \Longrightarrow D' = -\frac{p_E}{\eta_D Q_E}$$

$$\eta_S = -\frac{1}{S'}\frac{p_E}{Q_E} \Longrightarrow S' = \frac{p_E}{\eta_S Q_E}$$

Thus

$$p^{c\prime}(t_0) = \frac{D'}{D'-S'} = \frac{\eta_S}{\eta_D + \eta_S} > 0,$$

$$p^{s\prime}(t_0) = \frac{S'}{D'-S'} = -\frac{\eta_D}{\eta_D + \eta_S} < 0,$$

And

$$p^{c\prime}(t_0) - p^{s\prime}(t_0) = \frac{D'}{D'-S'} - \frac{S'}{D'-S'} = 1$$

--total burden is equal to the tax.

HW Supply-Demand Model with ad-valorem tax. If the tax is levied according to the price of the good, the model becomes

$$\begin{aligned} p^c &= D(Q) \\ p^s &= S(Q) \\ p^c &= p^s(1+t), \end{aligned}$$

$t = 0.07 = 7\%$

Recompute $Q'(t_0)$, $p^{c\prime}(t_0)$, and $p^{s\prime}(t_0)$.

HW Reformulate the model with specific tax using just two endogeneous variables Q and p^c. Apply Implicit Function Theorem to find $Q'(t_0)$, $p^{c\prime}(t_0)$, and $p^{s\prime}(t_0)$.

HW Reformulate the model with advalorem tax using just two endogeneous variables Q and p^s. Apply Implicit Function Theorem to find $Q'(t_0)$, $p^{c\prime}(t_0)$, and $p^{s\prime}(t_0)$.