Q3. In a duopoly market, suppose that firms 1 and 2 face market demand, $p = 100 - (q_1 + q_2)$. Firm costs are $c_1 = 10q_1$ and $c_2 = q_2^2$.

(a) Calculate market price under the Cournot environment. How much is the profit that each firm yields in the equilibrium?

(b) Calculate the market price under the joint production decision, i.e. collusive equilibrium. How much is the profit that each firm yields in the equilibrium?

(c) Do you think that the collusive equilibrium will sustain? What would be the profit of firm 1 if firm 1 chooses to deviate from the collusive allocation?

## Answer

3. a. Calculate market price under Cournot model. Find profit of each firm ($\pi_1$ & $\pi_2$)

Firm 1 :

$$\pi_1 = P \cdot q_1 - C_1$$
$$= [100 - (q_1 + q_2)]q_1 - 10q_1$$

F.O.C : $[q_1]$ : $\frac{\partial \pi_1}{\partial q_1} = 0 \iff 100 - 2q_1 - q_2 - 10 = 0$

$$q_1^* = \frac{90 - q_2}{2}$$

Firm 2 :

$$\pi_2 = P \cdot q_2 - C_2$$
$$= [100 - (q_1 + q_2)]q_2 - q_2^2$$

F.O.C : $[q_2]$ : $\frac{\partial \pi_2}{\partial q_2} = 0 \iff 100 - q_1 - 2q_2 - 2q_2 = 0$

$$q_2^* = \frac{100 - q_1}{4}$$

Solve for $q_1^*$ :

$$q_1^* = \frac{90 - \frac{100 - q_1}{4}}{2} = \frac{260 + q_1}{8}$$

$$8q_1 = 260 + q_1$$

$$7q_1 = 260 \Rightarrow q_1^* = \frac{260}{7} = 37.14 \text{ units}$$

$$q_2^* = \frac{100 - \frac{260}{7}}{4} = \frac{440}{28} = 15.71 \text{ units}$$

S.O.C : $H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$

$|H_1| = -2$

$|H_2| = (-2)(-4) - (-1)(-1) = 7$

$|H_1|$ & $|H_2|$ have alternate sign with negative as initial sign, and $\forall q_1$ & $\forall q_2$, so H is always negative definite.

Therefore, $q_1^*$ and $q_2^*$ from F.O.C is global maximum and $\pi_1$ & $\pi_2$ functions are globally concave.

Market price : $P = 100 - \left(\frac{260}{7} + \frac{440}{28}\right) = 100 - \left(\frac{260+110}{7}\right) = \frac{700-370}{7} = \frac{330}{7} = \boxed{47.14 \text{ units}}$

Profit for firm 1 : $\pi_1 = (47.14 \times 37.14) - (10 \times 37.14) = \boxed{1{,}379.59 \text{ units}}$

Profit for firm 2 : $\pi_2 = (47.14 \times 15.71) - 15.71^2 = \boxed{493.88 \text{ units.}}$

b. Calculate the market price and the profit that each firm gets under collusive equilibrium.

$$\pi^C = \pi_1 + \pi_2$$
$$= P \cdot q_1 + P \cdot q_2 - C_1 - C_2$$
$$= P(q_1 + q_2) - C_1 - C_2$$
$$= [100 - (q_1 + q_2)](q_1 + q_2) - 10q_1 - q_2^2$$
$$= 100(q_1 + q_2) - (q_1^2 + q_2^2 + 2q_1q_2) - 10q_1 - q_2^2$$

F.O.C : $[q_1]$ : $\frac{\partial \pi_c}{\partial q_1} = 0 \Leftrightarrow 100 - 2q_1 - 2q_2 - 10 = 0$

$$q_1^* = \frac{90 - 2q_2}{2}$$

$[q_2]$ : $\frac{\partial \pi_c}{\partial q_2} = 0 \Leftrightarrow 100 - 2q_2 - 2q_1 - 2q_2 = 0$

$$q_2^* = \frac{100 - 2q_1}{4}$$

Solve for $q_1^*$

$$q_1^* = \frac{90 - 2\left(\frac{100 - 2q_1}{4}\right)}{4} = \frac{90 - 50 + q_1}{4} = \frac{40 + q_1}{4}$$

$4q_1 = 40 + q_1$

$3q_1 = 40 \Rightarrow q_1^* = \frac{40}{3} = 13.33$ units

$q_2 = \frac{100 - 2\frac{40}{3}}{4} = \frac{220}{12} = 18.33$ units

S.O.C : $H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -4 \end{bmatrix}$

$|H_1| = -2$

$|H_2| = (-2)(-4) - (-2)(-2) = 4$

$|H_1|$ & $|H_2|$ have alternate sign with negative as initial sign, and $\forall q_1$ & $\forall q_2$, so H is always negative definite.

Therefore, $q_1^*$ and $q_2^*$ from F.O.C is global maximum and $\pi_1$ & $\pi_2$ functions are globally concave

Market price : $P = 100 - \left(\frac{40}{3} + \frac{220}{12}\right) = 100 - \left(\frac{40+55}{3}\right) = \frac{205}{3} = \boxed{68.33 \text{ units.}}$

Profit for firm 1: $\pi_1 = P \cdot q_1 - 10q_1 = \frac{205}{3} \cdot \frac{40}{3} - 10 \cdot \frac{40}{3} = \boxed{777.78 \text{ units.}}$

Profit for firm 2: $\pi_2 = P \cdot q_2 - q_2^2 = \frac{205}{3} \cdot \frac{220}{12} - \left(\frac{220}{12}\right)^2 = \boxed{916.67 \text{ units}}$

C. The collusion equilibrium will not sustain because if a firm does not commit or sign a contract, a firm deviates, or the authorities ban collusion, the merge will not occur.

If firm 1 deviates from collusion, its profit will be the same as before merging which is $\pi_1 = 1{,}379.59$ units.

Q4. Consider a simple two-market model where demand and supply for each market is given below. (Notationally, let's name the two markets as A and B, respectively.)

**Market A:**

Demand: $p_A = 10 - 2Q_A$

Supply: $p_A = 1 + Q_A$

**Market B:**

Demand: $p_B = 20 - Q_B$

Supply: $p_B = 2 + 2Q_B$

a. Derive the market equilibrium
b. Suppose the government imposes unit tax on both markets at the rate of $t_A$ and $t_B$. Solve for the after-tax equilibrium as the function of $t_A$ and $t_B$.
c. How much revenue can the government collect from the taxation?
d. Determine the level of $t_A$ and $t_B$ that maximizes government's revenue.

Answer

a. Derive the market equilibrium

Equilibrium for market A

$Q_A^d = Q_A^s$ or $P_A^d = P_A^s$

$$\Rightarrow \quad 10 - 2Q_A = 1 + Q_A$$

$$3Q_A = 9$$

$$Q_A^* = \frac{9}{3} = 3 \text{ units}$$

$$P_A^* = 1 + 3 = 4 \text{ units.}$$

Equilibrium for market B

$$P_B^D = P_B^S \Leftrightarrow 20 - Q_B = 2 + 2Q_B$$

$$3Q_B = 18$$

$$Q_B^* = \frac{18}{3} = 6 \text{ units}$$

$$\Rightarrow P_B^* = 2 + 2 \cdot 6 = 14 \text{ units.}$$

b. When government impose unit tax on both market :

Market A

$$P_A^s = P_A^d - t_A$$

$$\Rightarrow P_A^d - t = 1 + Q_A \Rightarrow P_A^d = 1 + Q_A + t_A$$

Equilibrium for market A: $P_A^s = P_A^d$

$$\Rightarrow \quad 1 + Q_A + t_A = 10 - 2Q_A$$

$$\Rightarrow \quad 3Q_A = 9 - t_A$$

$$\boxed{Q_A^* = 3 - \frac{t_A}{3}}$$

Supply $P_A^* = 1 + 3 - \frac{t_A}{3} = \boxed{4 - \frac{t_A}{3}}$ ; Demand $P_A^* = \boxed{4 - \frac{t_A}{3} + t_A}$

Market B

$$P_B^s = P_B^d - t_B$$

$$\Rightarrow \quad P_B^d - t_B = 2 + 2Q_B$$

$$P_B^d = 2 + 2Q_B + t_B$$

Equilibrium for market B : $P_B^s = P_B^d$

$$\Rightarrow \quad 2 + 2Q_B + t_B = 20 - Q_B$$

$$3Q_B = 18 - t_B$$

$$\Rightarrow \boxed{Q_B^* = 6 - \frac{t_B}{3}}$$

$\Rightarrow$ Demand $P_B^* = 20 - (6 - \frac{t_B}{3}) = \boxed{14 + \frac{t_B}{3}}$ ; Supply $P_B^* = 2 + 2(6 - \frac{t_B}{3}) = \boxed{14 - \frac{2}{3}t_B}$

c. Find the revenue of gov. from taxation

$$\Pi = \Pi_A + \Pi_B$$

$$= t_A \times Q_A^* + t_B \times Q_B^*$$

$$= t_A(3 - \frac{t_A}{3}) + t_B(6 - \frac{t_B}{3})$$

$$\boxed{\Pi. = 3t_A - \frac{t_A^2}{3} + 6t_B - \frac{t_B^2}{3}}$$

d. Determine the level of $t_A$ and $t_B$ that maximize gov.'s revenue.

F.O.C : $\frac{\partial \Pi}{\partial t_A} = 0 \iff 3 - \frac{2}{3}t_A = 0$

$$\boxed{t_A^* = \frac{9}{2}}$$

$\frac{\partial \Pi}{\partial t_B} = 0 \iff 6 - \frac{2}{3}t_B = 0$

$$\Rightarrow \boxed{t_B^* = \frac{18}{2} = 9}$$

S.O.C : $H = \begin{bmatrix} \Pi_{AA} & \Pi_{AB} \\ \Pi_{BA} & \Pi_{BB} \end{bmatrix} = \begin{bmatrix} -2/3 & 0 \\ 0 & -2/3 \end{bmatrix}$

$|H_1| = -2/3$

$|H_2| = (-2/3)(-2/3) - 0 = \frac{4}{9}$

Since $|H_1|$ & $|H_2|$ have alternate sign with initiative negative sign, H is negative definite. Also, for $\forall t_A$ & $\forall t_B$, H is always negative definite.

Therefore, $t_A^*$ and $t_B^*$ from F.O.C is global maximum and $\Pi$ function is globally concave.

Q5:

a). Construct the profit function :

$$Q = 2000 + 4\sqrt{A} - 20P$$

$$\Rightarrow P = 100 + \frac{1}{5}\sqrt{A} - \frac{1}{20}Q$$

$$C(Q,A) = c(Q) + A \; ; \quad c(Q) = 2Q + 1000$$

$$TC = 2Q + 1000 + A \; ;$$

$$\Pi = P \times Q - TC$$

$$\Pi = 100Q + \frac{1}{5}Q\sqrt{A} - \frac{1}{20}Q^2 - 2Q - 1000 - A$$

Profit function : $\Pi(Q,A) = \left(98 + \frac{1}{5}\sqrt{A}\right)Q - \frac{1}{20}Q^2 - A - 1000$

2). $\Pi = P \times Q - TC$

$$= 2000P + 4P\sqrt{A} - 20P^2 - 4000 - 8\sqrt{A} + 40P - 1000 - A$$

$$\Pi(P,A) = \left(2040 + 4\sqrt{A}\right)P - 20P^2 - 8\sqrt{A} - A - 5000$$

$$\frac{\partial \Pi}{\partial P} = 0 \Leftrightarrow 2040 + 4\sqrt{A} - 40P = 0 \quad - (1)$$

$$\frac{\partial \Pi}{\partial A} = 0 \Leftrightarrow \frac{4P}{2\sqrt{A}} - \frac{4}{\sqrt{A}} - 1 = 0 \quad - (2)$$

$$① \Rightarrow \quad 51 + \frac{1}{10}\sqrt{A} = P$$

$$② \Rightarrow \quad 2P - 4 - \sqrt{A} = 0 \Rightarrow \sqrt{A} = 2P - 4$$

$$51 + \frac{1}{10}(2P-4) = P$$

$$\Rightarrow 0.8P = 51 - \frac{2}{5}$$

$$P = 63.25\$$$

$$\sqrt{A} = 126.5 - 4 \Rightarrow A = 15006.25\$$$

To maximize the profit of the monopolist:

$$\boxed{P^* = 63.25\$, \; A^* = 15006.25\$}$$

c). Check for definiteness of the Hession Matrix:

$$\Pi_P = 51 + \frac{1}{10}\sqrt{A} - P \; ; \; \Pi_A = 2P - 4 - \sqrt{A}$$

$$H = \begin{bmatrix} \Pi_{PP} & \Pi_{PA} \\ \Pi_{AP} & \Pi_{AA} \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{20\sqrt{A}} \\ 2 & -\frac{1}{2\sqrt{A}} \end{bmatrix}$$

$$P = 63.25\$ \; ; \; A = 15006.25\$$$

$$H = \begin{bmatrix} -1 & 1/2450 \\ 2 & -1/245 \end{bmatrix} ; |H_1| = -1 ; |H_2| = (-1)(-1/245) - (2)(1/2450)$$

$$= \frac{10}{2450} - \frac{2}{2450} = \frac{8}{2450}$$

+ $|H_1|$ negative ; $|H_2|$ positive.

$\Rightarrow$ at $(P = 63.25; A = 15006.25)$ ; $d^2\Pi < 0 \Rightarrow \Pi$ is concave $\Rightarrow$ this point is local max

Q1. Given the production function $Q = f(K, L) = A[K^n + L^n]$ where A is the level of technology, K is capital and L is labor. Suppose that $n > 0$. Consider the following problem.

a. To ensure that the above production function exhibits a *decreasing return to scale technology*, what additional restrictions do one need to place on $n$?

b. Under the assumption used in (a), show that the production function satisfies the law of diminishing returns.

c. Calculate the marginal rate of technical substitution (MRTS) of labor (L) for capital (K).

d. Show that MRTS is a decreasing function in L. That is, as labor increases, the value of MRTS decreases.

e. Under the condition(s) assumed in (a), is the production function satisfied with the *global* concave property?

Suppose that $K(t) = \frac{1}{2}t^2 + 2t + 3$ and $L(t) = e^t + 3$, where $t \geq 0$ is the number of periods from now. Consider the following problem

f. Show that Q is increasing over time.

g. Compute $\frac{\frac{dQ}{Q}}{dt}$ when t = 0, i.e. growth of output in the initial period.

a. Decreasing return to scale on technology : $0 < \Delta Q < \Delta A$

if $n = 1 \Rightarrow Q = A[K^1 + L^1]$

increase input by 2 times $Q_1 = A[(2K)^1 + (2L)^1]$

$= A[2^1(K^1 + L^1)]$

$Q_1 = 2A[K^1 + L^1]$

$Q_1 = 2Q$ constant return to scale

if $n = 0.4 \Rightarrow Q = A[K^{0.5} + L^{0.5}]$

increase input by 2 times $Q_1 = A[(2K)^{0.5} + (2L)^{0.5}]$

$= 2^{0.5} A[K^{0.5} + L^{0.5}]$

$Q_1 \approx 1.41 Q$ decreasing return to scale

Thus $\underline{0 < n < 1}$

b. if $n = 0.5$

increase input by 3 $\Rightarrow Q_2 = 3^{0.5} Q$

$= 1.73 Q$

thus increasing additional input by 2 times yield $Q_1 = 1.41 Q$ or additional $0.41 Q$

increasing additional input by 3 time yield $Q_2 = 1.73Q$ or additional $1.73Q - 1.41Q$

$= 0.32 Q$

$\Rightarrow$ return decreases with additional input

$\rightarrow$ production function satisfies law of diminishing return.

c. Calculate MRTS of L for K

$$MRTS = \left.\frac{dK}{dL}\right|_{Q=Q_0} \quad , \quad f(K, L) = A[K^n + L^n]$$

$$= AK^n + AL^n$$

$$\left.\frac{dK}{dL}\right|_{Q=Q_0} = -\frac{F_L}{F_K} = -\frac{nAL^{n-1}}{nAK^{n-1}} = \boxed{\frac{L^{n-1}}{K^{n-1}}}$$

d. Show that MRTS is a decreasing function in L

$$\frac{\partial MRTS}{\partial L} = \frac{1}{K^{n-1}} \cdot (n-1) L^{n-2} = \frac{n-1}{K^{n-1}} L^{n-2}$$

since $n-1 < 0$ and $L^{n-2} > 0$

$\rightarrow \frac{\partial MRTS}{\partial L} < 0$ thus MRTS decreases when L increases

e. $Q = A[K^n + L^n]$ ; $0 < n < 1$

F.O.C: $[K]$ : $\frac{\partial Q}{\partial K} = 0 \Leftrightarrow nAK^{n-1} = 0$

$[L]$ : $\frac{\partial Q}{\partial L} = 0 \Leftrightarrow nAL^{n-1} = 0$

S.O.C $H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix} = \begin{bmatrix} nA(n-1)K^{n-2} & 0 \\ 0 & nA(n-1)L^{n-2} \end{bmatrix}$

$|H_1| = nA(n-1)K^{n-2}$; since $n-1 < 0 \Rightarrow |H_1| < 0$

$|H_2| = nA(n-1)K^{n-2} \cdot nA(n-1)L^{n-2} - 0 = nA(n-1)K^{n-2} \cdot nA(n-1)L^{n-2}$ since $nA(n-1)K^{n-2}$ and $nA(n-1)L^{n-2}$ are both negative $\Rightarrow |H_2| > 0$

And for any $K, L$ in condition $0 < n < 1$, $|H_1|$ always $< 0$ & $|H_2|$ always $> 0$

So, $H$ is always negative definite for all $(K, L)$. Therefore, production function is globally concave.

f. show that $Q$ is increasing overtime

$$dQ = \frac{\partial Q}{\partial K} \cdot \frac{\partial K}{\partial t} dt + \frac{\partial Q}{\partial L} \cdot \frac{\partial L}{\partial t} dt$$

$$= nAK^{n-1} \cdot (t+2)\, dt + nAL^{n-1} \cdot (e^t)\, dt$$

$$\frac{dQ}{dt} = nAK^{n-1}(t+2) + nAL^{(n-1)}(e^t) > 0$$

$\Rightarrow$ Q increases overtime.

g. compute $\dfrac{\frac{dQ}{Q}}{dt}$ when $t = 0$

when $t = 0 \Rightarrow K = 3, L = 4, Q = A[3^n + 4^n]$

$$\frac{dQ}{dt} \cdot \frac{1}{Q} = \frac{nAK^{n-1}(t+2) + nAL^{n-1}e^t}{A(K^n + L^n)} = \frac{nA(3)^{n-1}(2) + nA(4)^{n-1}(1)}{A3^n + A4^n}$$

Q2. A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

a. What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

Since $Q = K^{1/3} L^{2/3}$, if K, L increase by 1 units

$Q = 1^{1/3 + \frac{2}{3}} (K^{1/3} L^{2/3})$

$= 1Q$ ⇒ constant return to scale.

substitute $c=2$

b. Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$\Pi = P.Q - WL - Kr \Rightarrow \Pi = \frac{1}{(K^{1/3}L^{2/3})^{c}} \cdot K^{1/3}L^{2/3} - WL - Kr = K^{-\frac{c+1}{3}} L^{-\frac{2c+2}{3}} - WL - Kr = K^{-1/3} L^{-2/3} - WL - Kr$$

c. The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

$$Q = K^{1/3} L^{2/3}$$

$$\frac{\partial \Pi}{\partial K} = -\frac{1}{3} K^{-4/3} L^{-2/3} = r \quad \text{—— } (1)$$

$$\frac{\partial \Pi}{\partial L} = -\frac{2}{3} K^{-1/3} L^{-5/3} = W \quad \text{—— } (2)$$

$$K^{1/3} = \frac{Q}{L^{2/3}} \quad \text{substitute in } (2)$$

$$= -\frac{2}{3}\left(\frac{L^{2/3}}{Q}\right) \cdot L^{-5/3} = W$$

$$= -\frac{2L^{-1}}{3Q} = W$$

$$= -\frac{2}{3WQ} = L^{*}$$

$$L^{2/3} = \frac{Q}{K^{1/3}} \quad \text{substitute in } (1)$$

$$= -\frac{1}{3} K^{-4/3} \left(\frac{K^{1/3}}{Q}\right) = r$$

$$= -\frac{K^{-1}}{3Q} = r$$

$$K^{*} = -\frac{1}{3Qr}$$

d. How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

$$L^* = -\frac{2}{3wQ} = -\frac{2w^{-1}}{3Q}$$

$$\frac{\partial L}{\partial w} = \frac{2w^{-2}}{3Q} > 0$$

positive relationship where the increase in wage will cause demand for labor to increase

$$\frac{\partial L}{\partial r} = 0$$

the increase in renting doesn't affect demand for labor

e. Confirm your answer with the second-order condition.

$$\frac{\partial \Pi}{\partial K} = -\frac{1}{3}K^{-4/3}L^{-2/3} = r \quad \text{—— } 1$$

$$\frac{\partial \Pi}{\partial L} = -\frac{2}{3}K^{-1/3}L^{-5/3} = w \quad \text{—— } 2$$

$$H = \begin{bmatrix} \Pi_{KK} & \Pi_{KL} \\ \Pi_{LK} & \Pi_{LL} \end{bmatrix} = \begin{bmatrix} \frac{4}{9}K^{-7/3}L^{-2/3} & \frac{2}{9}K^{-4/3}L^{-5/3} \\ \frac{2}{9}K^{-\frac{4}{3}}L^{-\frac{5}{3}} & \frac{10}{9}K^{-1/3}L^{-\frac{8}{3}} \end{bmatrix}$$

$$|H_1| = \frac{4}{9}K^{-7/3}L^{-2/3} > 0$$

$$|H_2| = \frac{40}{81}K^{-8/3}L^{-10/3} - \frac{4}{81}K^{-8/3}L^{-10/3}$$

$$= \frac{36}{81}K^{-8/3}L^{-10/3} - \frac{41}{81}K^{-\frac{8}{3}}L^{-\frac{10}{3}} > 0$$

Since $|H_1|$ and $|H_2|$ has positive sign so $H$ is positive definite for all $L, K$ so it is globally minimixer where $\Pi$ function is globally convex