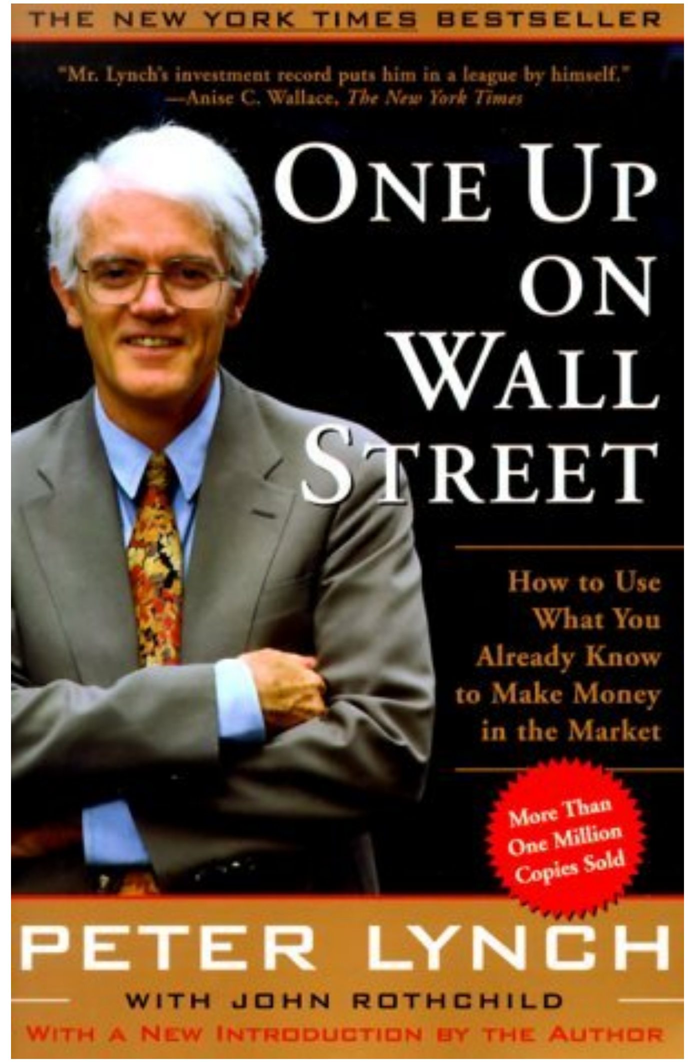
Portfolio Theory I : Two Assets

FN 201: Business Finance

Recommended Finance book

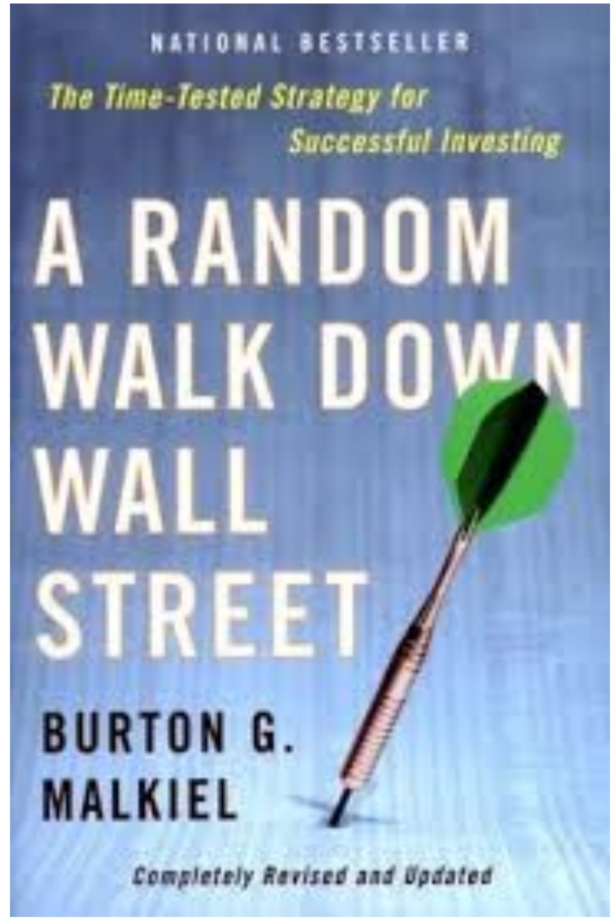
More than one million copies have been sold of this seminal book on investing in which legendary mutual-fund manager Peter Lynch explains the advantages that average investors have over professionals and how they can use these advantages to achieve financial success.



Read more: http://www.businessinsider.com/the-most-important-finance-books-2013-12?op=1#ixzz3UzfHifdY

Recommended Finance book

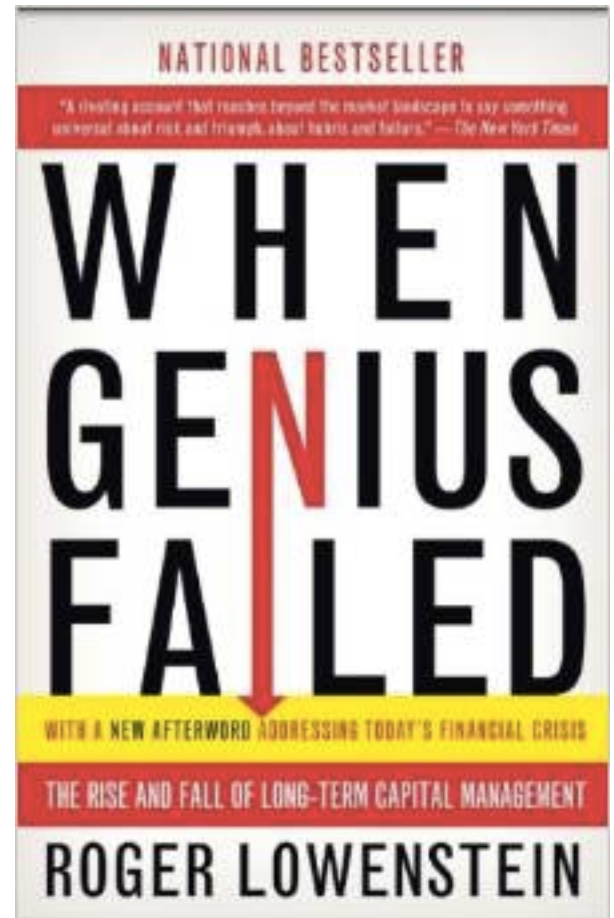
In a time of market volatility and economic uncertainty, when high-frequency traders and hedge fund managers seem to tower over the average investor, Burton G. Malkiel's classic and gimmick-free investment guide is now more necessary than ever. Rather than tricks, what you'll find here is a time-tested and thoroughly research-based strategy for your portfolio. Whether you're considering your first 401(k) contribution or contemplating retirement, this fully updated edition of A Random Walk Down Wall Street should be the first book on your reading list.



http://www.amazon.com/Random-Walk-Down-Wall-Street/dp/0393246116/ref=sr_1_1?s=books&ie=UTF8&qid=1426914511&sr=1-1&keywords=a+random+walk+down+wall+street

Recommended Finance book

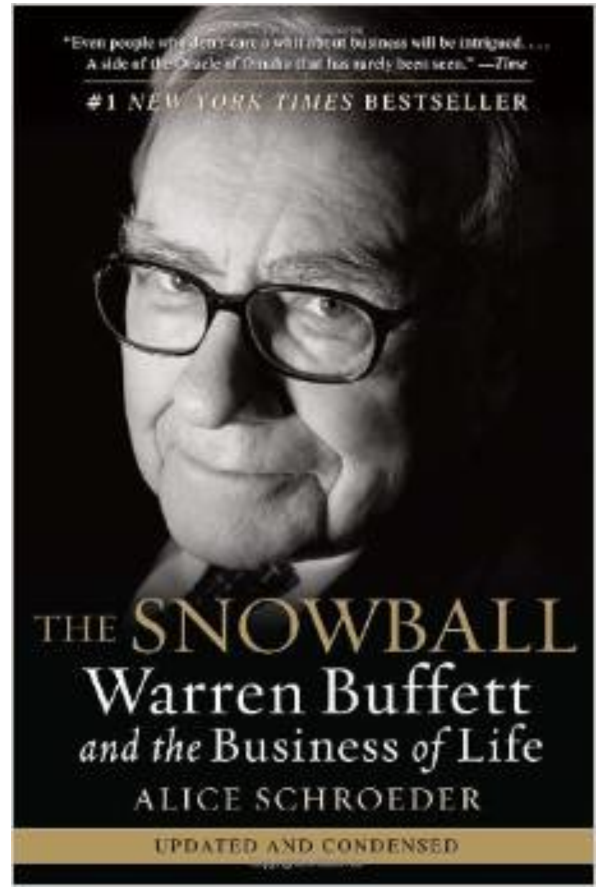
In this business classic—now with a new Afterword in which the author draws parallels to the recent financial crisis—Roger Lowenstein captures the gripping roller-coaster ride of Long-Term Capital Management. Drawing on confidential internal memos and interviews with dozens of key players, Lowenstein explains not just how the fund made and lost its money but also how the personalities of Long-Term's partners, the arrogance of their mathematical certainties, and the culture of Wall Street itself contributed to both their rise and their fall.



http://www.amazon.com/When-Genius-Failed-Long-Term-Management/dp/0375758259/ref=sr_1_1?s=books&ie=UTF8&qid=1426914675&sr=1-1&keywords=when+genius+fail

Recommended Finance book

Here is THE book recounting the life and times of one of the most respected men in the world, Warren Buffett. The legendary Omaha investor has never written a memoir, but now he has allowed one writer, Alice Schroeder, unprecedented access to explore directly with him and with those closest to him his work, opinions, struggles, triumphs, follies, and wisdom. The result is the personally revealing and complete biography of the man known everywhere as “The Oracle of Omaha.”



http://www.amazon.com/Snowball-Warren-Buffett-Business-Life/dp/0553384619/ref=sr_1_1?s=books&ie=UTF8&qid=1426914929&sr=1-1&keywords=snow+ball+warren+buffet

Learning objectives

- How to calculate the expected return and standard deviation of portfolios
- How to combine risky and risk-free assets
- How to find the most efficient combination of stocks into a portfolio?

Overview

- Two-asset portfolio math
- Risk aversion
- Choice between risky and risk-free asset
- Sharpe ratio and the Capital Allocation Line
- Choice between two risky assets
- Efficient frontier
- Tangency portfolio

Motivation

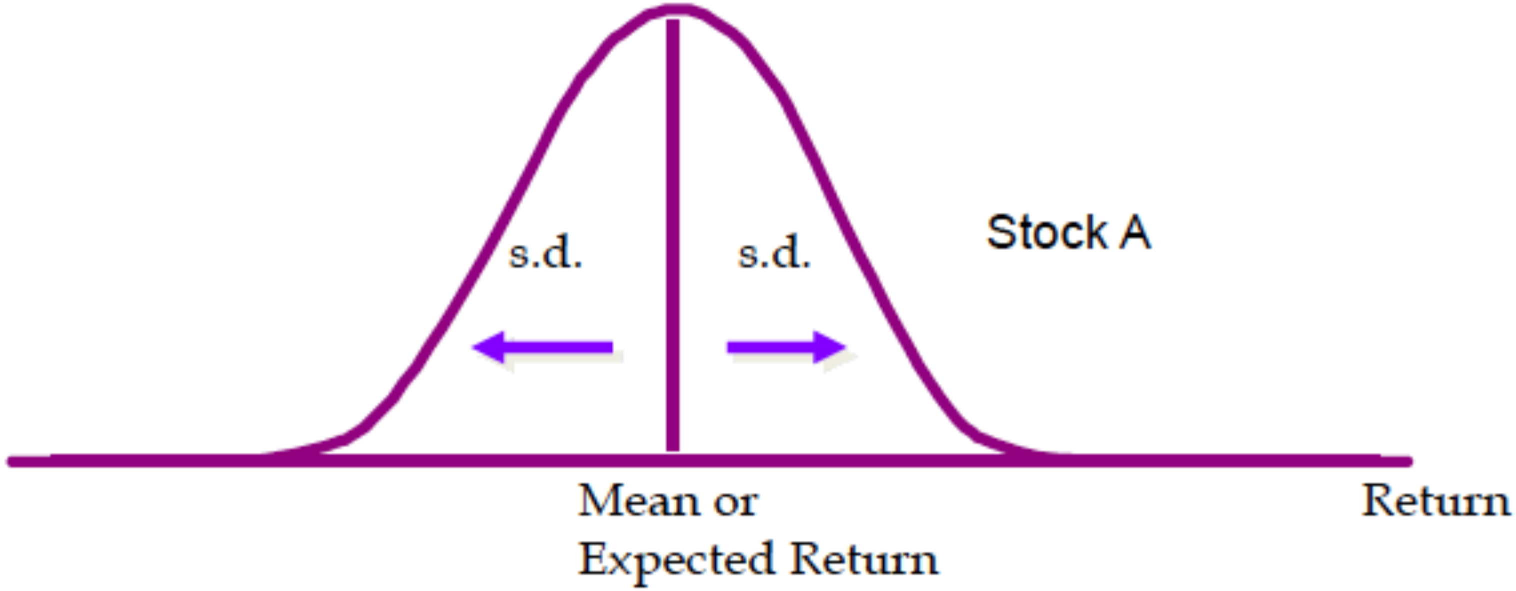
- Mean-variance portfolio analysis
  - Developed by Harry Markowitz in the early 1960's (1990 Nobel Prize in Economics)
  - One of the foundations of modern finance
- Used by all pension plans, endowments, wealthy individuals, banks, insurance companies, ...
- There is an industry of advisors (e.g. Wilshire Associates) and software makers (e.g. BARRA, Quantal) that implement what we will learn in these couple of classes

Portfolio Theory

- Markowitz showed exactly how an investor can reduce the s.d. of portfolio return by diversification
- If we look a histogram of daily returns of most stocks we observe that they are (close to) normally distributed
- The normal distribution can completely be described by two numbers:
  - Average, mean or “expected return”
  - Standard deviation of return

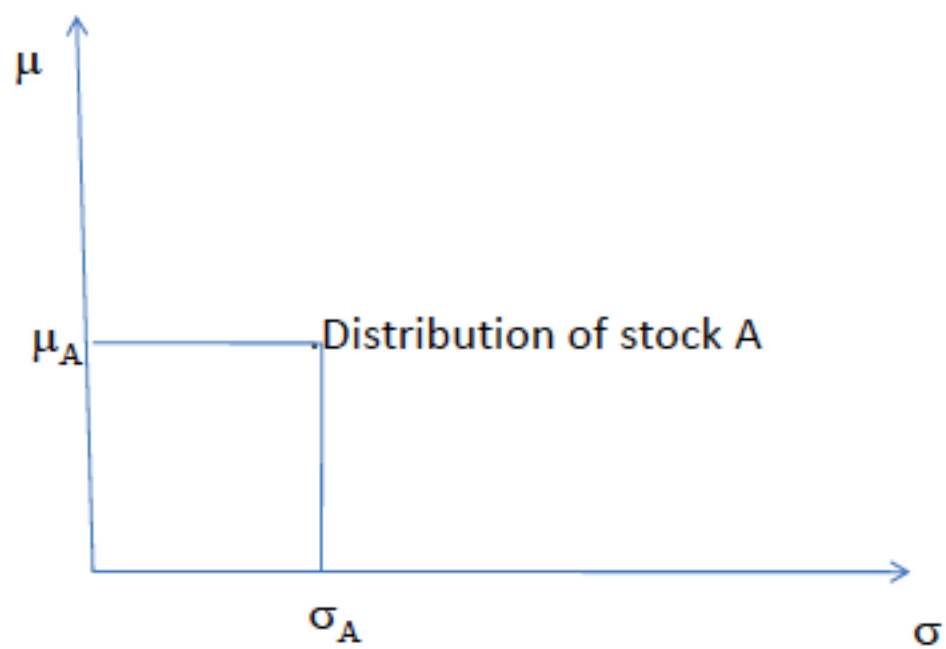
Probability Distribution of Returns

- Normal Distribution is completely described by
- Mean : Expected return
- Variance (standard deviation) : dispersion



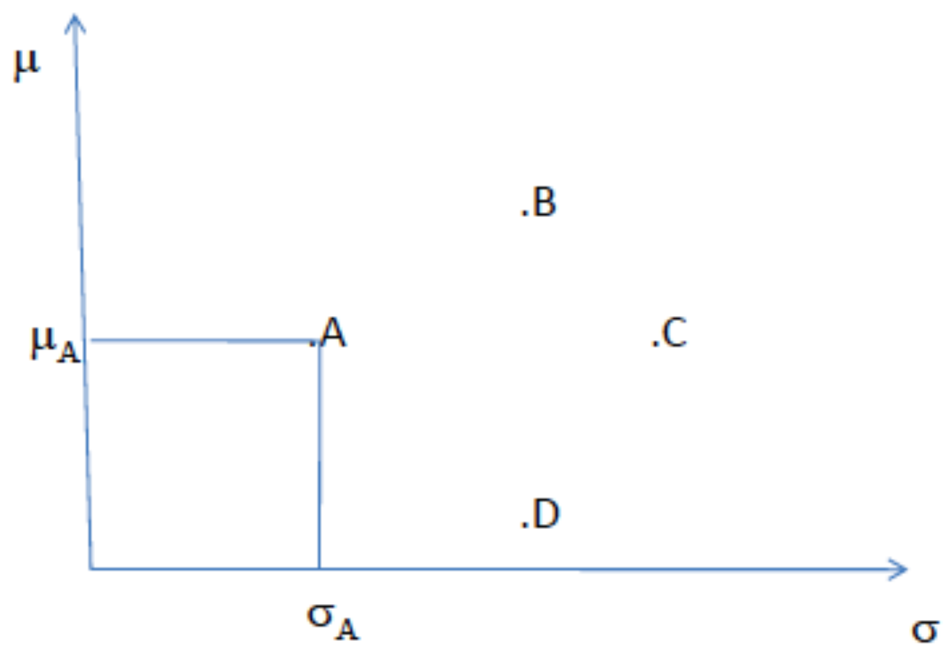
Distribution can be represented by one point in the graph

- Expected return (μ) vs. standard deviation (σ)



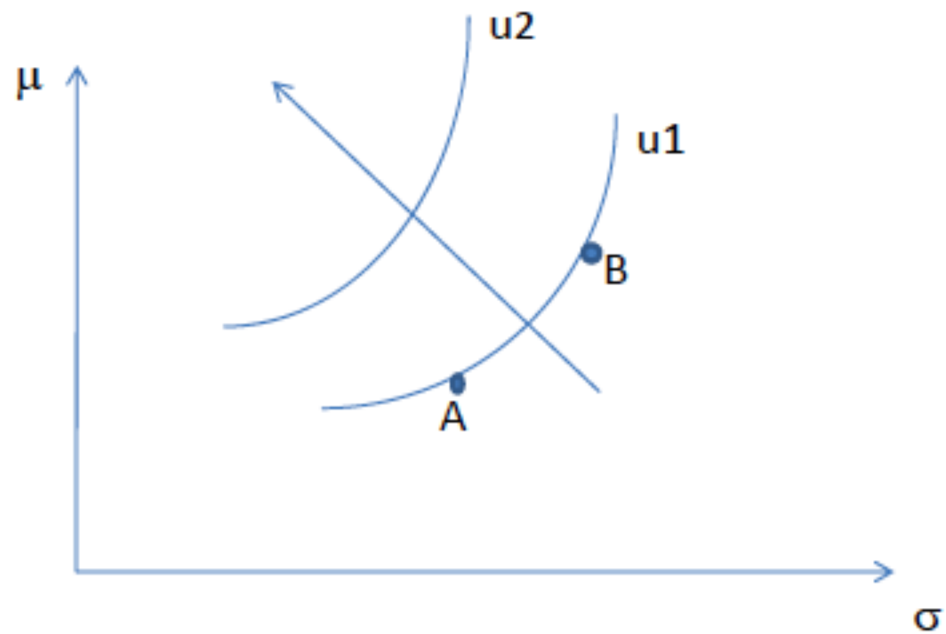
If you can only choose one stock?

- Most investors like expected return and dislike standard deviation (risk aversion)



Indifference Curves (for risk averse)

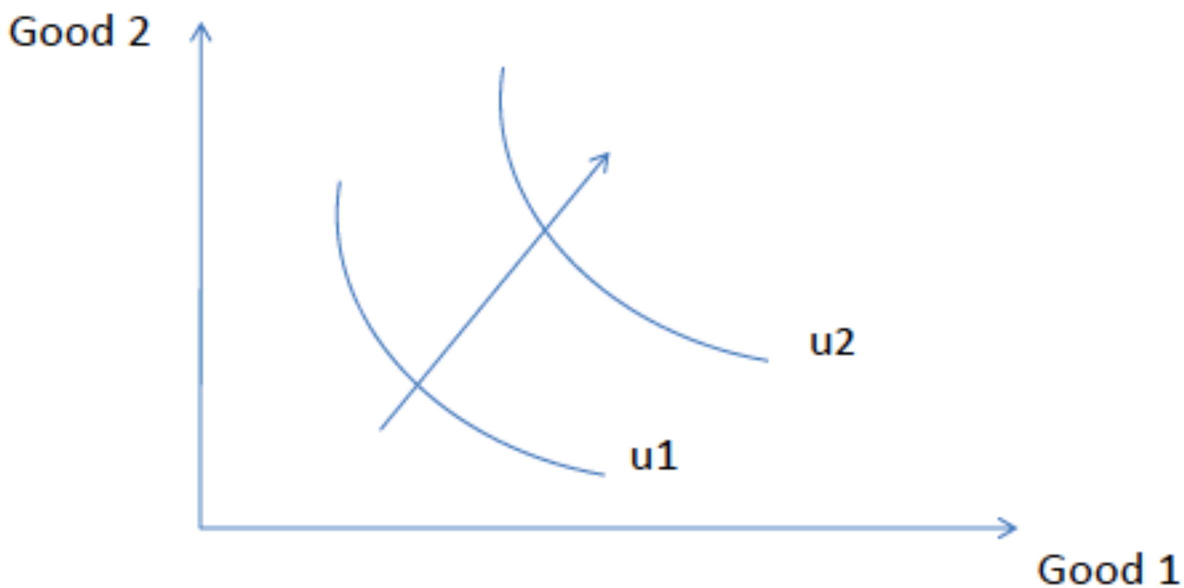
The choice between A and B depends on your preferences



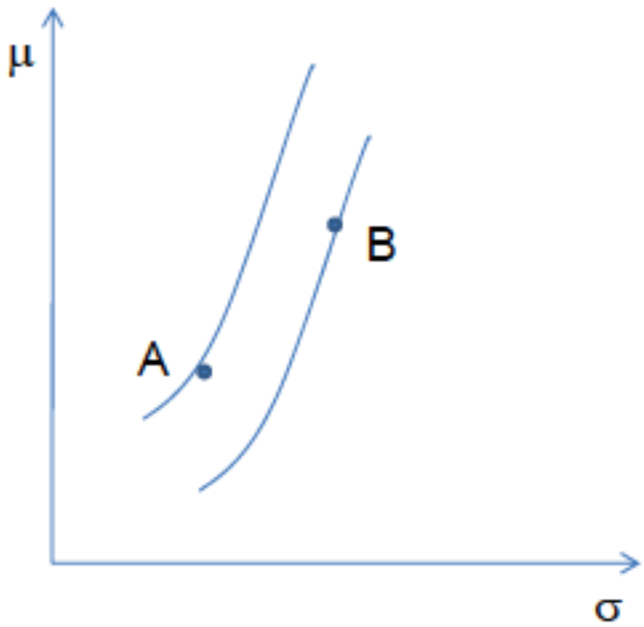
But, there is no reason to restrict your portfolio to holding one security, you could buy a combination of both (portfolio)

Indifference Curves in Economics

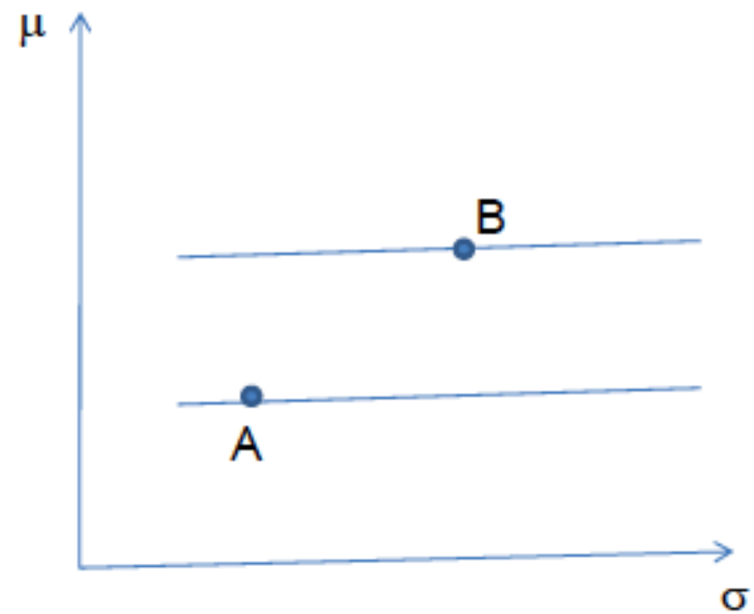
In this case both goods are desirable



Indifference Curves



Very risk averse (steep)



Risk neutral

Four-step approach to portfolio optimization

1. Estimate expected returns, standard deviations of returns, and correlations between assets
2. Find the optimal portfolio of one risk free asset and one risky asset
3. Find the optimal portfolio of risky assets (like stocks and bonds) – the optimal risky portfolio
4. Combine the optimal risky portfolio with the risk-free asset (T-bills) according to your risk aversion

Portfolio weights

- Fraction of wealth invested in different assets
  - Fractions
  - Add up to 1.0
  - Usually denoted by ‘w’
- Example
  - $100 in savings account at bank, $200 in IBM
    - Total investment: $100+$200=$300
  - Portfolio weights
    - Savings account: $100/$300 = 1/3
    - IBM: $200/$300 = 2/3
- Can we have negative portfolio weights?
  - $600 in IBM, Borrow $300 (weights: 2, -1)

Notation

- Portfolio weights

  w_x and w_y (with $w_x+w_y=1.0$)
- Expected (mean) returns

  $\mu_x=E(r_x)$ and $\mu_y=E(r_y)$
- Variances of returns (standard deviations)

  $\sigma^2{}_x=Var(r_x)$ and $\sigma^2{}_y=Var(r_y)$
- Covariance of returns

  $\sigma_{xy} = Cov(r_x , r_y) = \rho_{xy}\,\sigma_x\,\sigma_y$

Portfolio return and expected return

- Portfolio return (random)
  - Dollar value at end of period (plus cash flows) divided by dollar value at beginning of period
  - Can be computed as average of returns on individual securities weighted by their portfolio weights

$$r_p = w_x r_x + w_y r_y$$

- Then the expected return on the portfolio is

$$\mu_p = w_x \mu_x + w_y \mu_y$$

Remember from stats that $E(aX+bY)=aE(X)+bE(Y)$

Expected Return

- $\mu_x=0.10$
- $\mu_y=0.20$
- $w_x=0.5$
- $w_y=0.5$
- $\mu_p=0.5 \times 0.1 + 0.5 \times 0.2 = 0.15$

Portfolio expected return and variance

- The expected return is a weighted average of the expected return on the assets in the portfolio.
- But, the variance is not a weighted average. It depends on how the returns on the assets in the portfolio *covary (correlation, covariance)*.
- Diversification can reduce the variance of a portfolio (extreme example: life insurance company).

Covariance and Correlation

Covariance

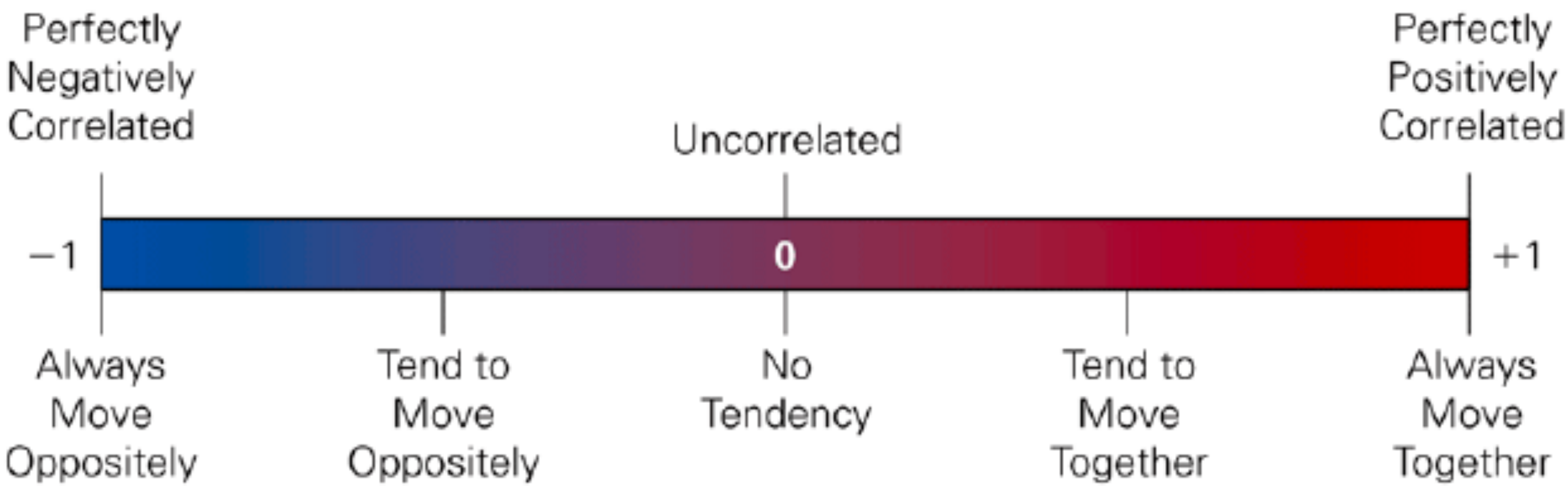
$$\sigma_{xy} = \rho_{xy}\,\sigma_x\,\sigma_y$$

Correlation

$$\rho_{xy} \begin{cases} 1 & \text{perfectly correlated} \\ 0 & \text{uncorrelated} \\ -1 & \text{perfectly negatively correlated} \end{cases}$$

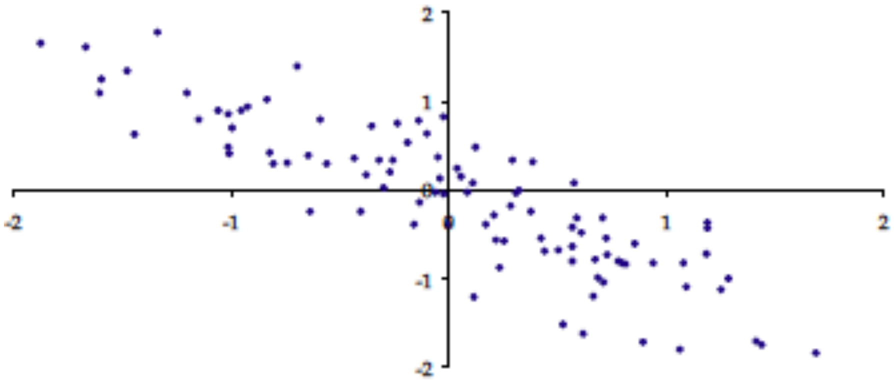
Covariance with itself = Variance ➡ $\sigma_{xx} = \rho_{xx}\,\sigma_x\,\sigma_x = \sigma^2_x$

Correlation

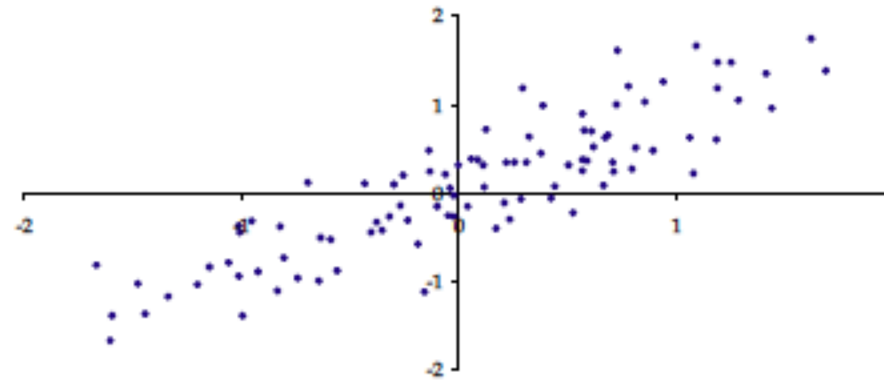


Correlations

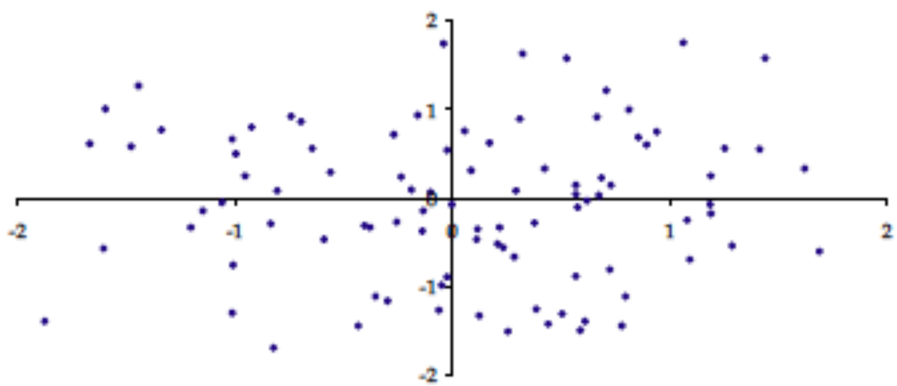
Correlation=-0.9



Correlation=+0.9



Correlation=0.0



Historical Annual Volatilities and Correlations for Selected Stocks

	Microsoft	Dell	Alaska Air	Southwest Airlines	Ford Motor	General Motors	General Mills
Volatility (Standard Deviation)	37%	50%	38%	31%	42%	41%	18%
Correlation with							
Microsoft	1.00	0.62	0.25	0.23	0.26	0.23	0.10
Dell	0.62	1.00	0.19	0.21	0.31	0.28	0.07
Alaska Air	0.25	0.19	1.00	0.30	0.16	0.13	0.11
Southwest Airlines	0.23	0.21	0.30	1.00	0.25	0.22	0.20
Ford Motor	0.26	0.31	0.16	0.25	1.00	0.62	0.07
General Motors	0.23	0.28	0.13	0.22	0.62	1.00	0.02
General Mills	0.10	0.07	0.11	0.20	0.07	0.02	1.00

Portfolio variance

- The variance of a portfolio is

$$\sigma^2_p = w_x^2 \sigma^2_x + w_y^2 \sigma^2_y + 2 w_x w_y \sigma_{xy}$$

Rem $Var(aX+bY)=a^2Var(X)+b^2Var(Y)+2abCov(X,Y)$

Covariance matrix

Portfolio Weights	w_x	w_y
w_x	$cov(r_x,r_x)$	$cov(r_x,r_y)$
w_y	$cov(r_x,r_y)$	$cov(r_y,r_y)$

Border-multiplied covariance matrix

Portfolio Weights	w_x	w_y
w_x	$w_x w_x cov(r_x,r_x)$	$w_x w_y cov(r_x,r_y)$
w_y	$w_y w_x cov(r_y,r_x)$	$w_y w_y cov(r_y,r_y)$

Variance

- $\sigma^2{}_p = w_x{}^2\,\sigma^2{}_x + w_y{}^2\,\sigma^2{}_y + 2w_x\,w_y\,\sigma_{xy}$
- σ_x=0.10
- σ_y=0.30
- ρ = 0.6
- σ_{xy}=0.1 x 0.3 x 0.6

One risky asset and one risk-free asset

- Individual assets
  - Portfolio weights w in risky asset (stocks) and (1-w) in risk-free asset (T-bill)
  - Expected (mean) returns : μ=7.5% and r_f =1.5%
  - Standard deviation of returns : σ =20% and 0
  - Covariance of returns : 0   ($\sigma_{xy} = \rho_{xy} \sigma_x \sigma_y$)
- Portfolio expected return

  $\mu_p = w \mu + (1\text{-}w) r_f = r_f + w(\mu - r_f) = 0.015 + 0.06w$
- Portfolio variance $\sigma_p^2 = w^2 \sigma^2 = 0.04w^2$

  Standard deviation $\sigma_p = w\sigma = 0.2w$

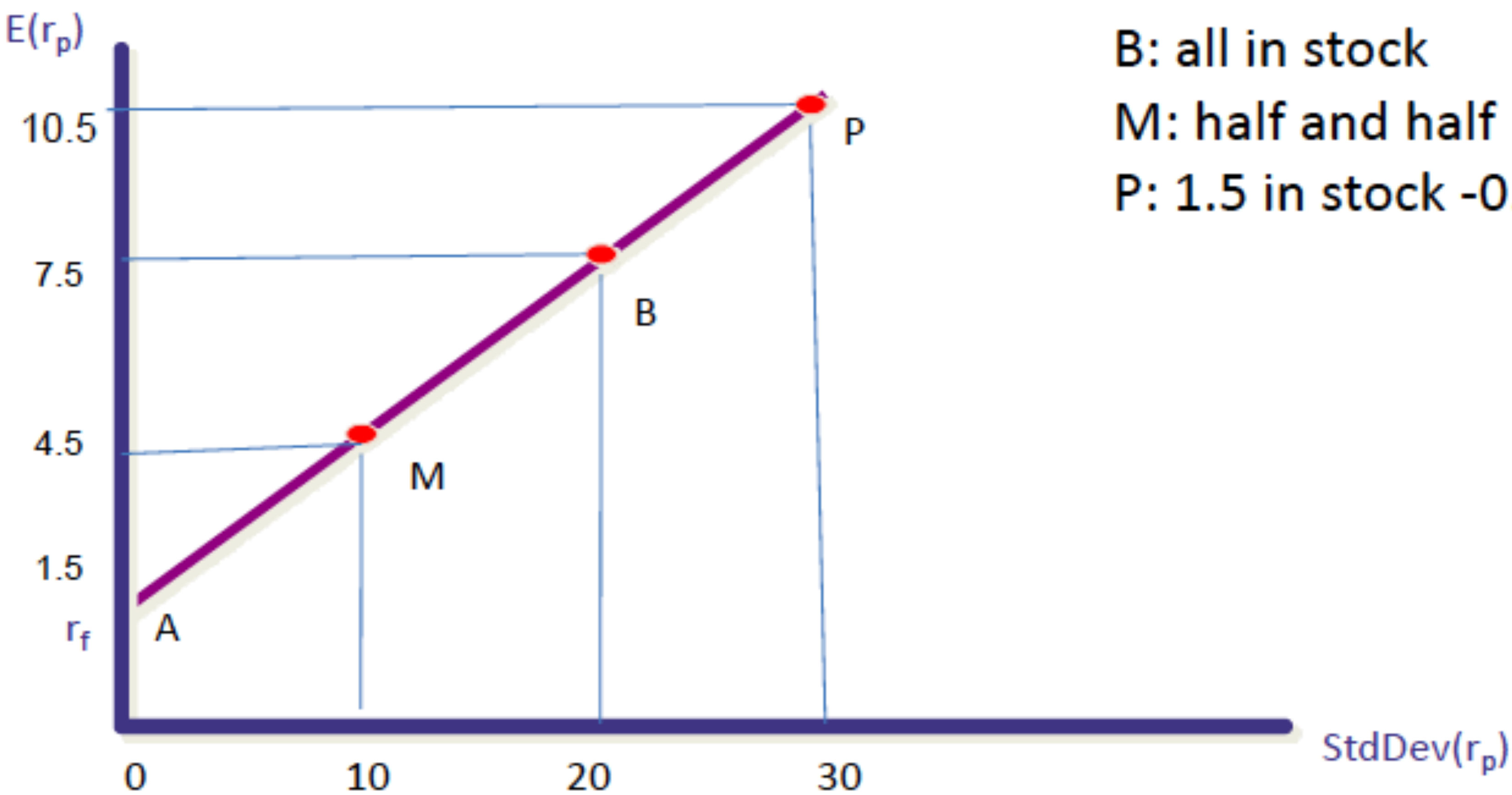
Different weights in the stock and the risk free asset

Portfolio	Stock	Risk free A	Exp. Ret. μ_p	S.D. σ_p
A	0	1	1.5%	0
B	1	0	7.5%	20%
M	0.5	0.5	4.5%	10%
P	1.5	-0.5	10.5%	30%

$$\mu_p = w\,\mu + (1-w)\,r_f = 0.015 + 0.06w$$

$$\sigma_p = w\sigma = 0.2w$$

One risky asset and one risk-free asset



A: all in risk free asset

B: all in stock

M: half and half

P: 1.5 in stock -0.5 in r_f

One risky asset and one risk-free asset

- Lending
  - Invest in the risk free asset (long position)
  - Take a positive position in the risk free asset
- Borrowing
  - Short the risk free asset (short position)
  - Take a negative position in the risk free asset
- Short a stock
  - Borrow a stock and sell it
  - Take a negative position in the stock

Short Selling

- Short selling involves selling securities you do not own
- Your broker borrows the securities from another client and sells them in the market in the usual way
- Having a negative position in the security (long: positive position)

Short Selling

- At some stage you must buy the securities back so they can be replaced in the account of the client
- You must pay dividends and other benefits the owner of the securities receives
- You benefit when the security price goes down

Capital Allocation Line

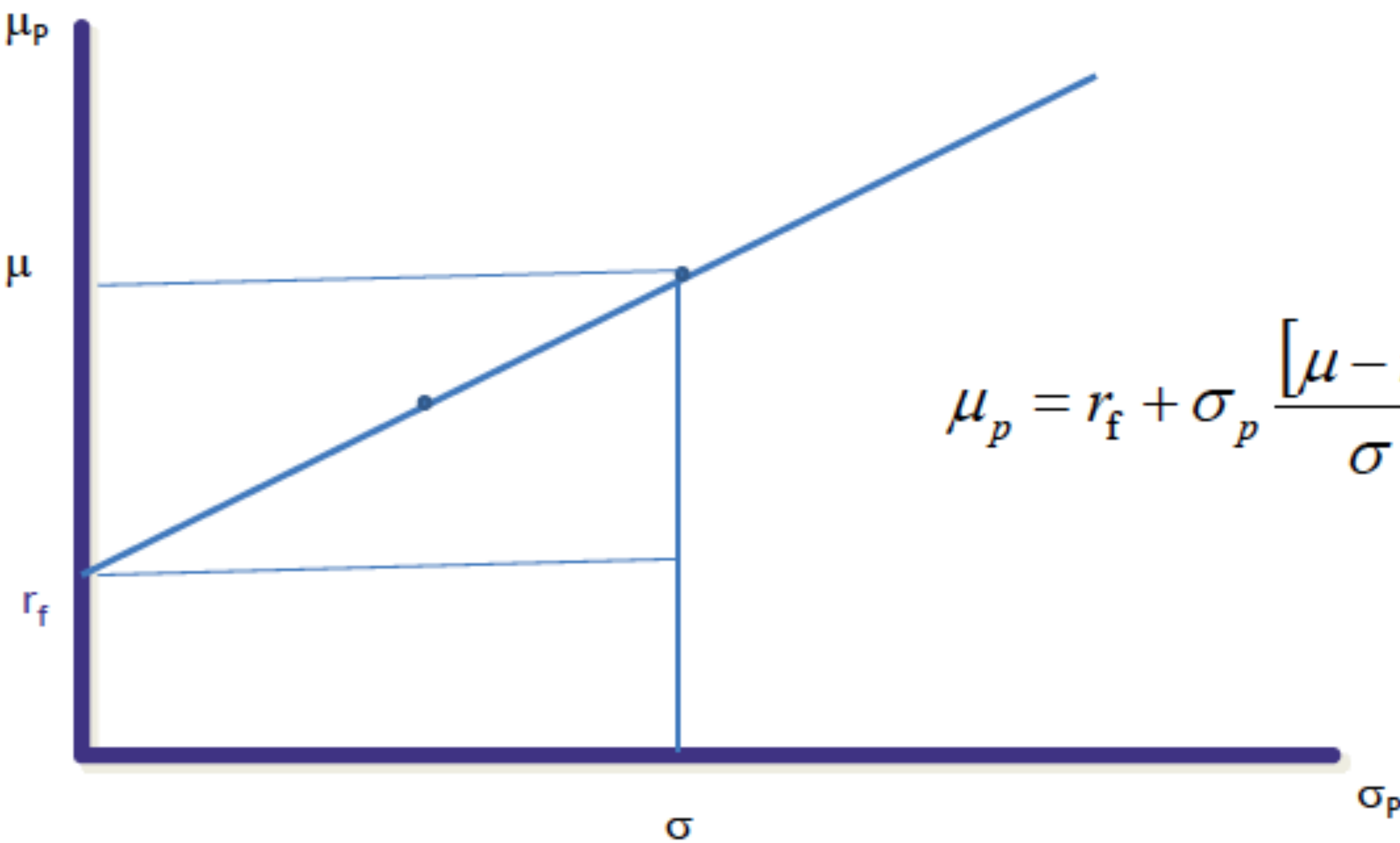
- Feasible combinations of mean and standard deviation – Capital Allocation Line

$$\mu_p = r_{\mathrm{f}} + \sigma_p \frac{[\mu - r_{\mathrm{f}}]}{\sigma} = 0.015 + 0.3\sigma_p$$

- Slope of CAL is called the Sharpe ratio: excess return per unit of risk

$$S = \frac{\mu - r_{\mathrm{f}}}{\sigma} = \frac{0.06}{0.2} = 0.3$$

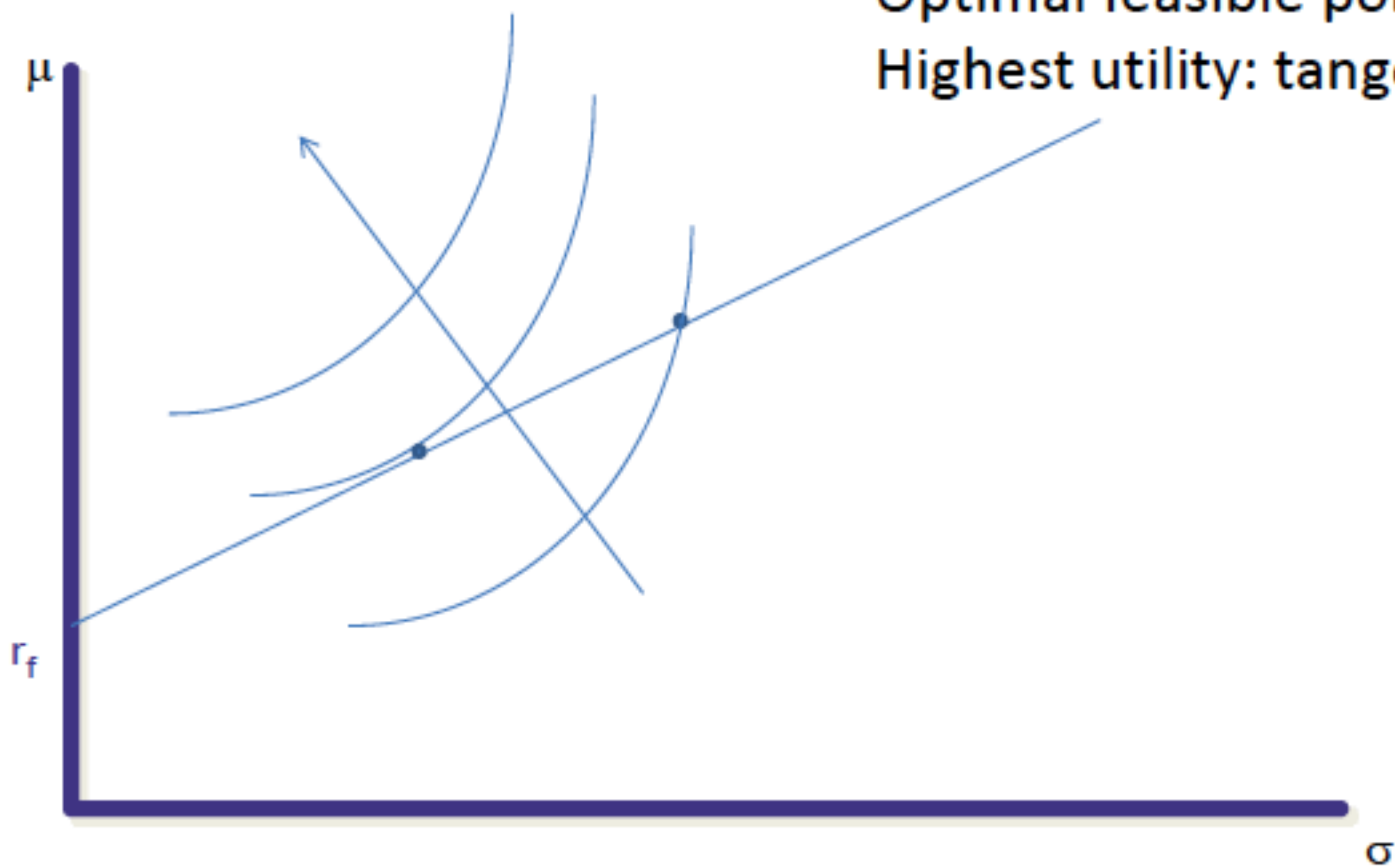
Capital Allocation Line



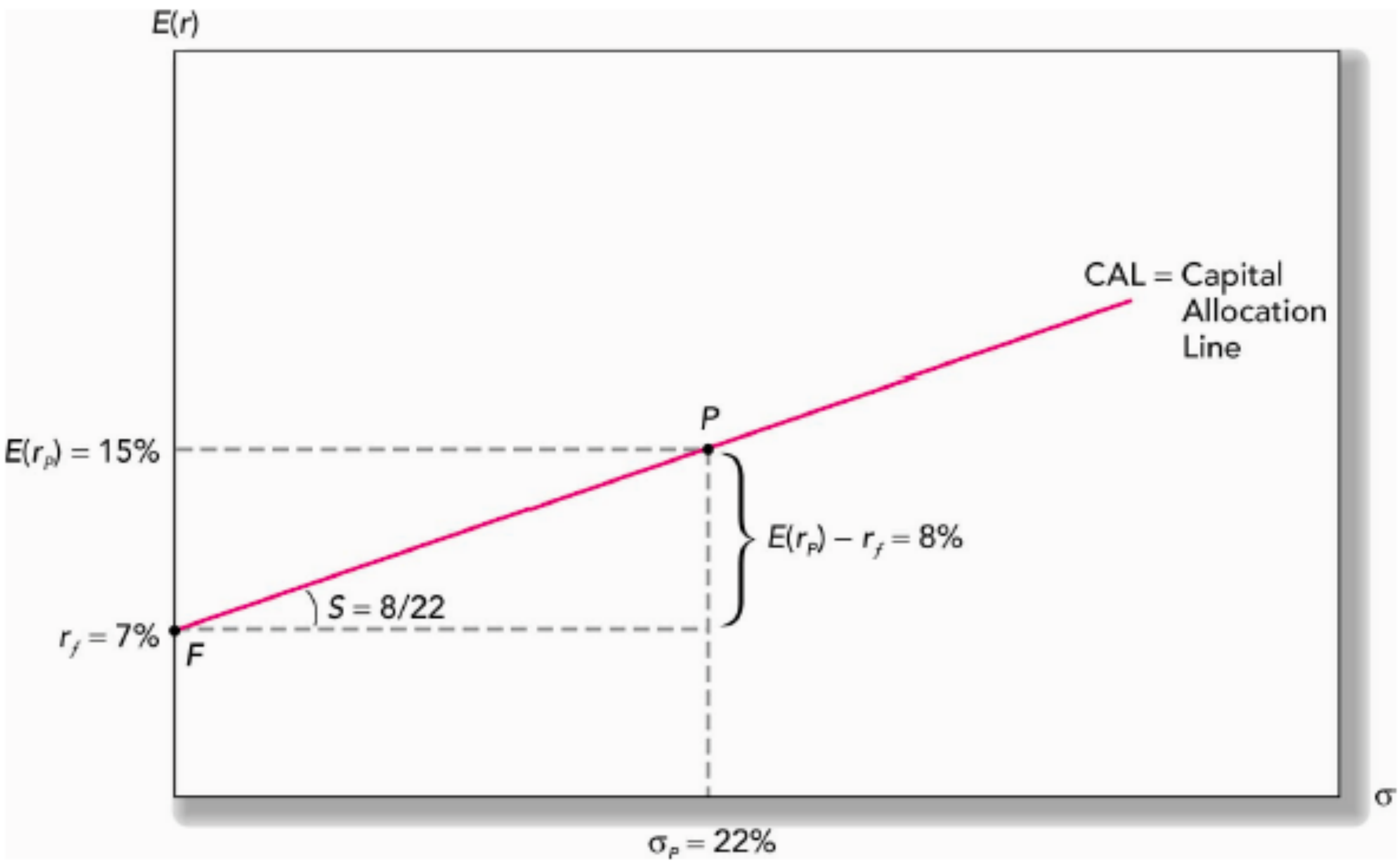
$$\mu_p = r_f + \sigma_p \frac{[\mu - r_f]}{\sigma}$$

Optimal Portfolio on the CAL

Optimal feasible portfolio
Highest utility: tangency point



The Investment Opportunity Set with a Risky Asset and a Risk-free Asset in the Expected Return-Standard Deviation Plane



Two risky assets

- We now know how to optimally combine one risky asset with a risk-free asset
- What if we have several risky assets?
  - First combine the risky assets into an optimal risky portfolio
  - Then treat the optimal risky portfolio as if it was a single asset and combine it with the risk-free asset
- Let's start with two risky assets

Two risky assets

- Two risky assets – bond and stock portfolios
  - Means of 3.0% and 7.5%
  - Standard deviations of 10% and 20%
  - Correlation of 0.2
- Consider a portfolio with 1/3 of funds invested in bond and 2/3 of funds invested in stock portfolios

Two risky assets

- Portfolio expected return

  $\mu_p = 1/3 \times 0.03 + 2/3 \times 0.075$

  $= 0.06 = 6.0\%$

- Portfolio variance= $w_x{}^2 \sigma^2{}_x + w_y{}^2 \sigma^2{}_y + 2 w_x w_y \sigma_{xy}$ $\sigma_{xy} = \rho \sigma_x \sigma_y$

  $\sigma_p{}^2 = (1/3)^2 (0.1)^2 + (2/3)^2 (0.2)^2 + 2\,(1/3)(2/3)(0.2 \times 0.1 \times 0.2)$

  $= 0.0207$

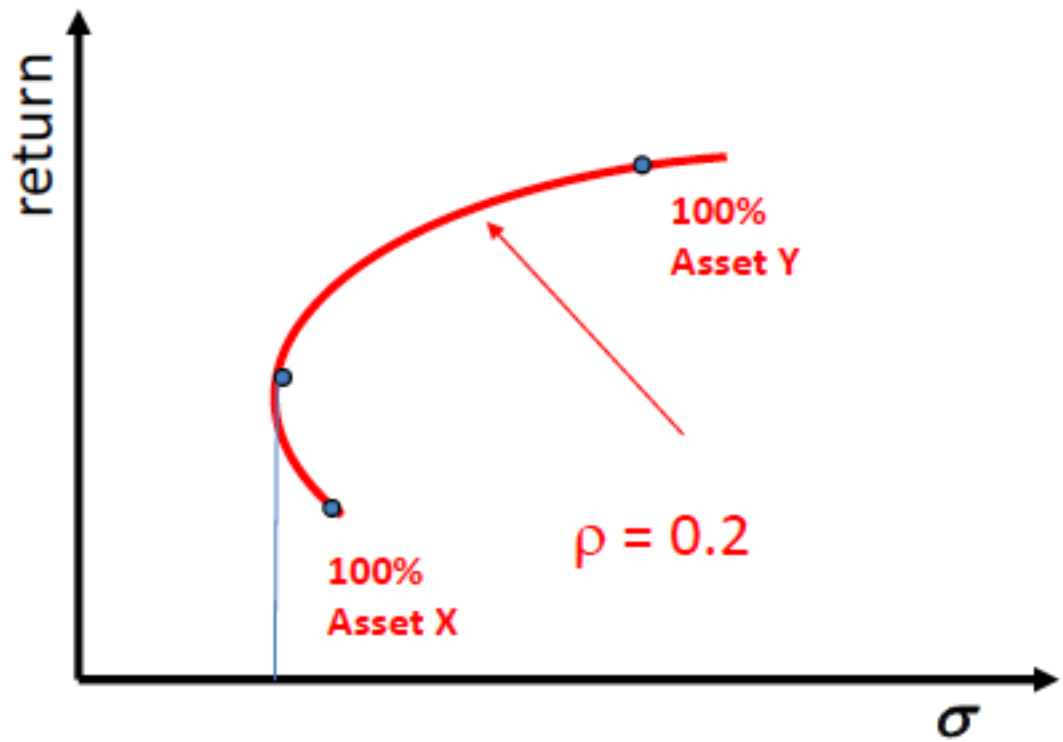
- Portfolio standard deviation

  $\sigma_p = \sqrt{0.0207}$

  $= 0.1438 = 14.38\%$

- What happens if the proportions change?
- What happens when the correlation changes?

Two risky assets when proportions change



Efficient Portfolios: upper part of the curve

Perfect Correlation?

$$\sigma_P^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \rho\sigma_x\sigma_y$$

$$\rho = 1$$

$$\sigma_P^2 = (w_x\sigma_x + w_y\sigma_y)^2$$

$$\sigma_P = w_x\sigma_x + w_y\sigma_y$$

$$\rho = -1$$

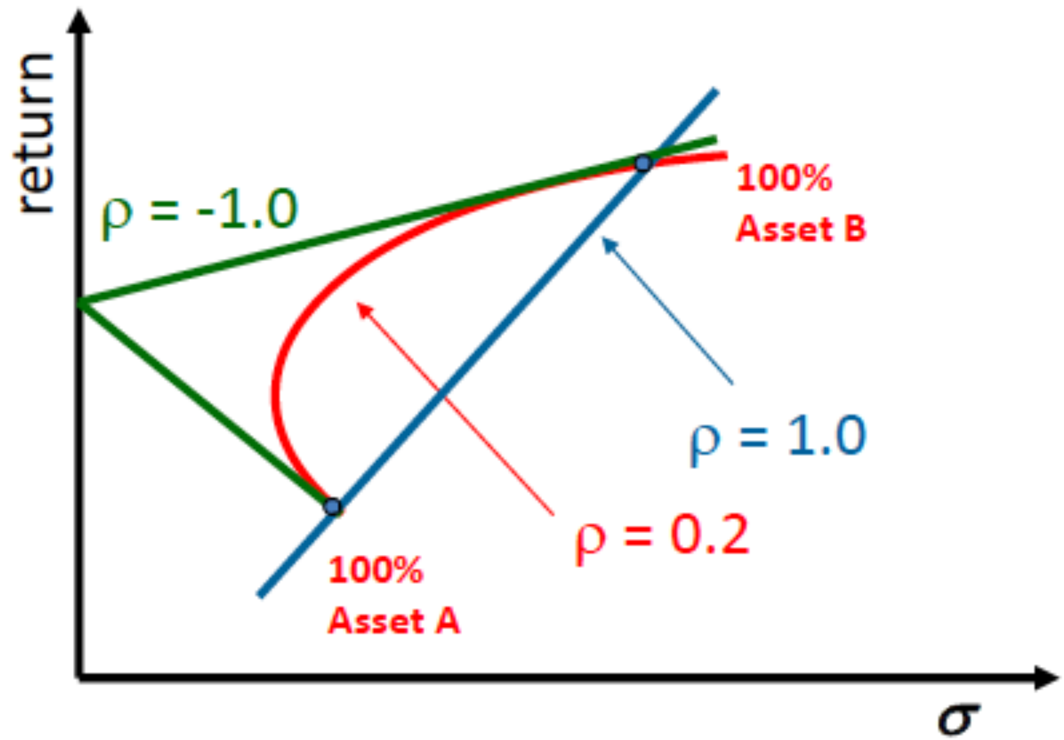
$$\sigma_P^2 = (w_x\sigma_x - w_y\sigma_y)^2$$

$$\sigma_P = w_x\sigma_x - w_y\sigma_y > 0$$

$$\sigma_P = w_y\sigma_y - w_x\sigma_x > 0$$

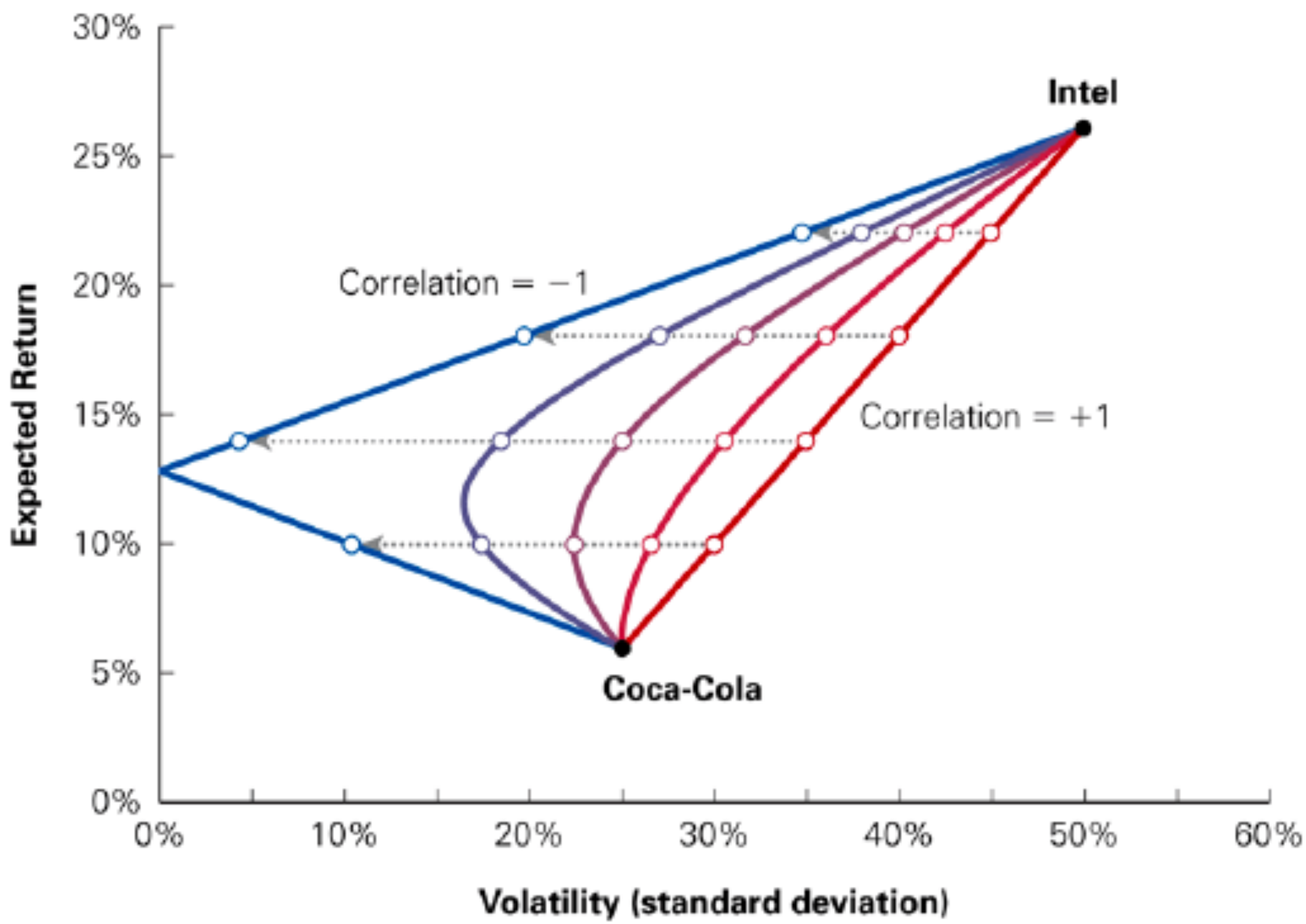
$$\sigma_P = 0 \ \ when \quad \frac{w_x}{w_y} = \frac{\sigma_y}{\sigma_x}$$

Two-Assets: Different Correlations

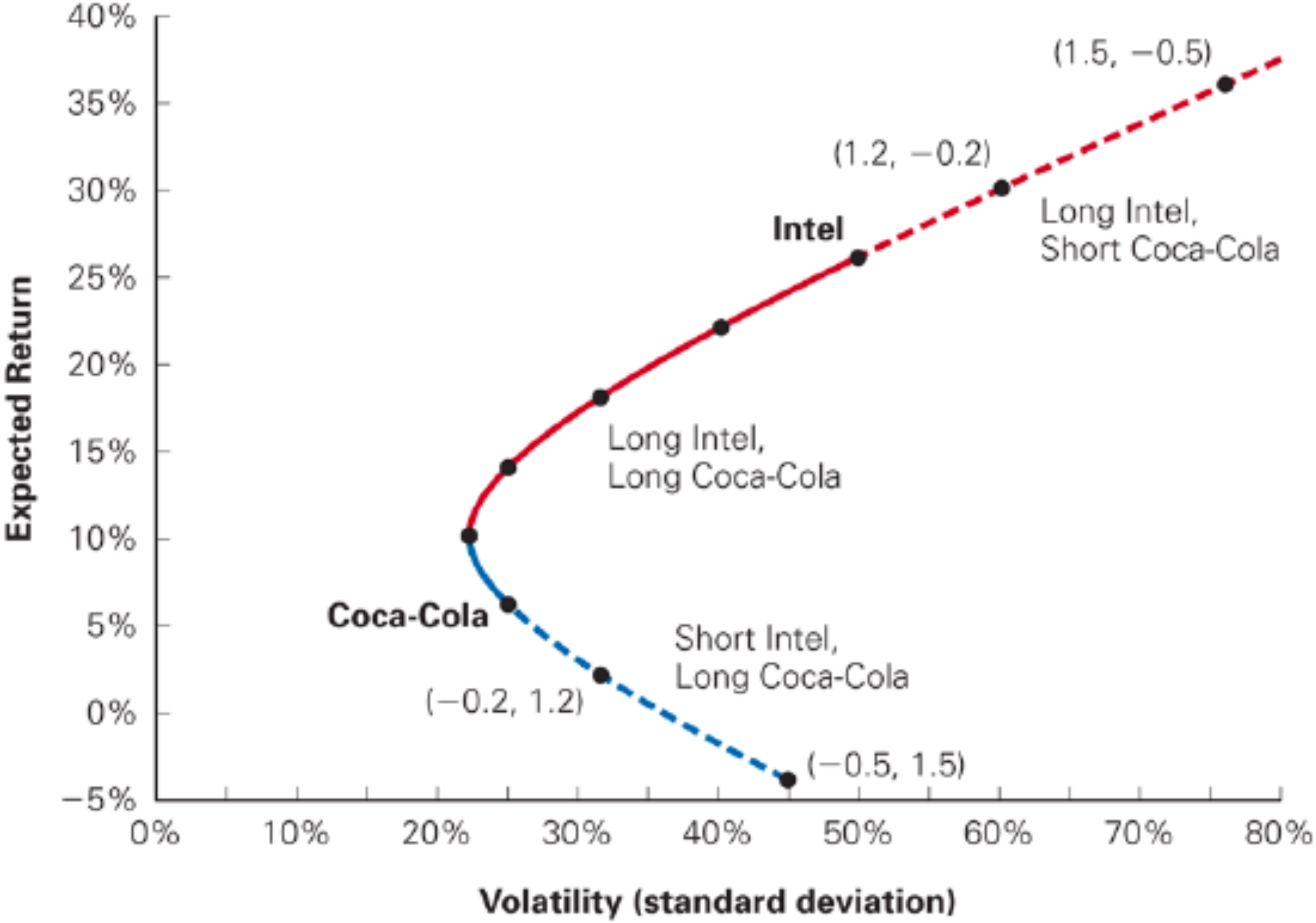


Efficient Portfolios: highest expected return for a given σ

Effect on Volatility and Expected Return of Changing the Correlation between Intel and Coca-Cola Stock



Portfolios of Intel and Coca-Cola Allowing for Short Sales



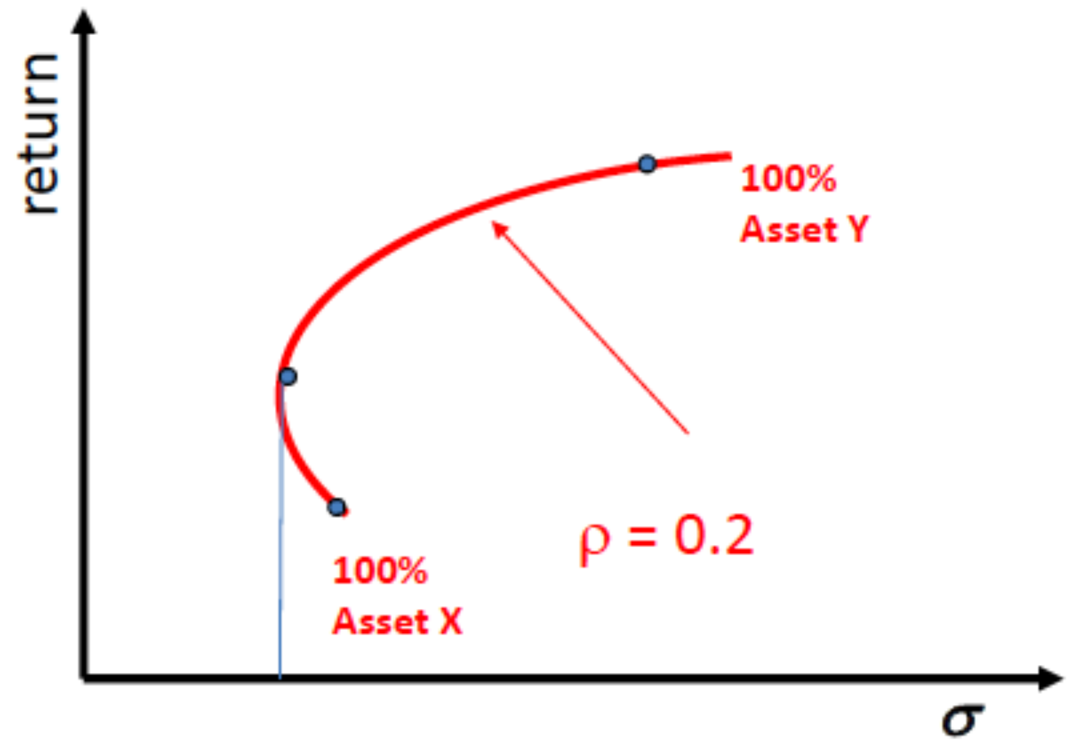
Two Assets: Correlation Effects

- Relationship depends on correlation coefficient
- $-1.0 \leq \rho \leq +1.0$
- The smaller the correlation, the greater the risk reduction potential
- If $\rho = +1.0$, no risk reduction is possible

Correlation and Diversification

- The various combinations of risk and return available all fall on a smooth curve.
- This curve is called an *investment opportunity set* because it shows the possible combinations of risk and return available from portfolios of these two assets.
- A portfolio that offers the highest return for its level of risk is said to be an *efficient portfolio*.
- The undesirable portfolios are said to be *dominated* or *inefficient*.

Minimum Variance Portfolio



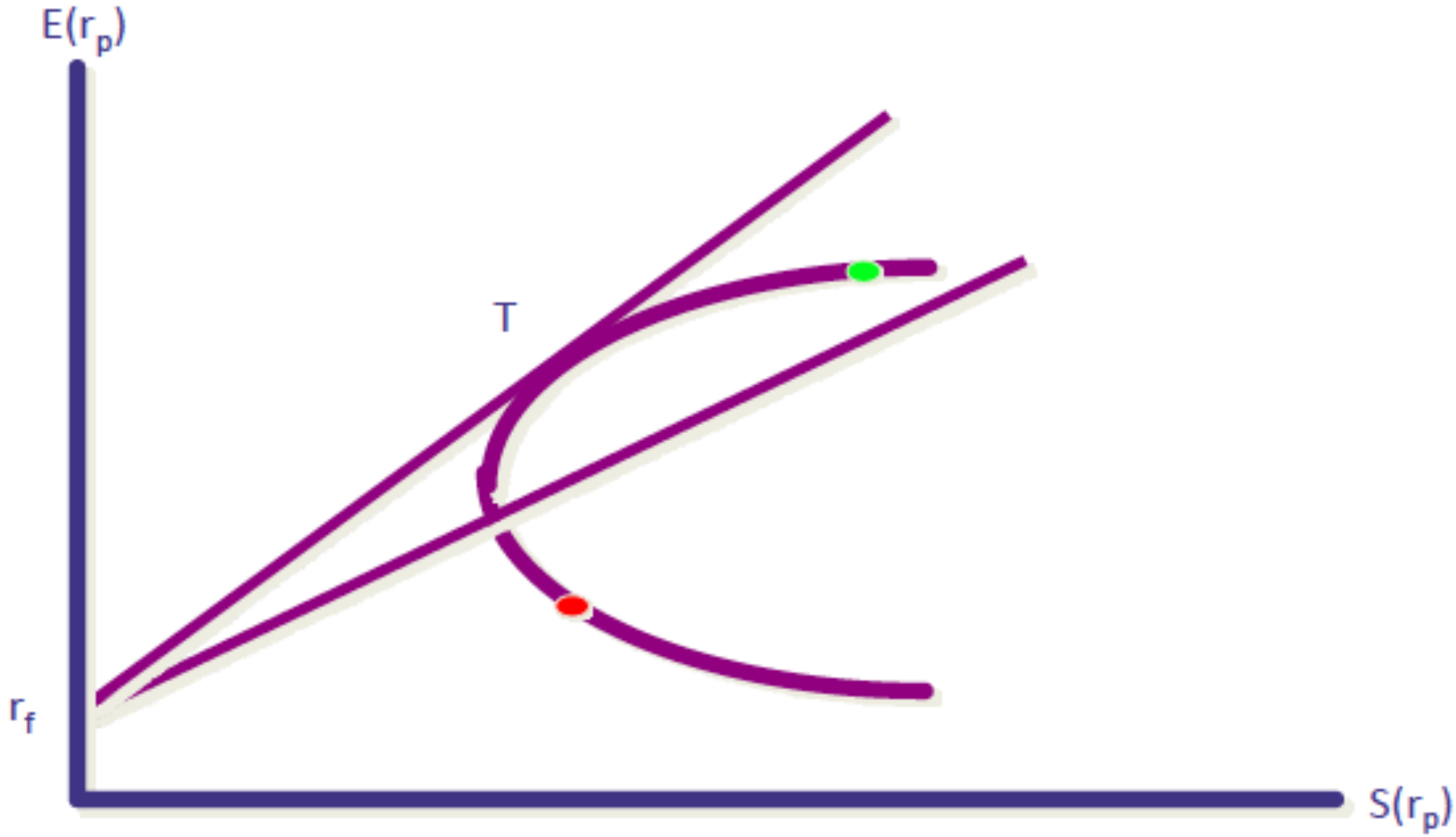
Efficient Portfolios: upper part of the curve

Two risky and one risk-free asset

- We now find the portfolio of the two risky assets that can be optimally combined with the risk-free asset
  - This is the *optimal risky portfolio*
- For each risky portfolio, find out the corresponding CAL
- What is the CAL that would give the investor maximum utility?

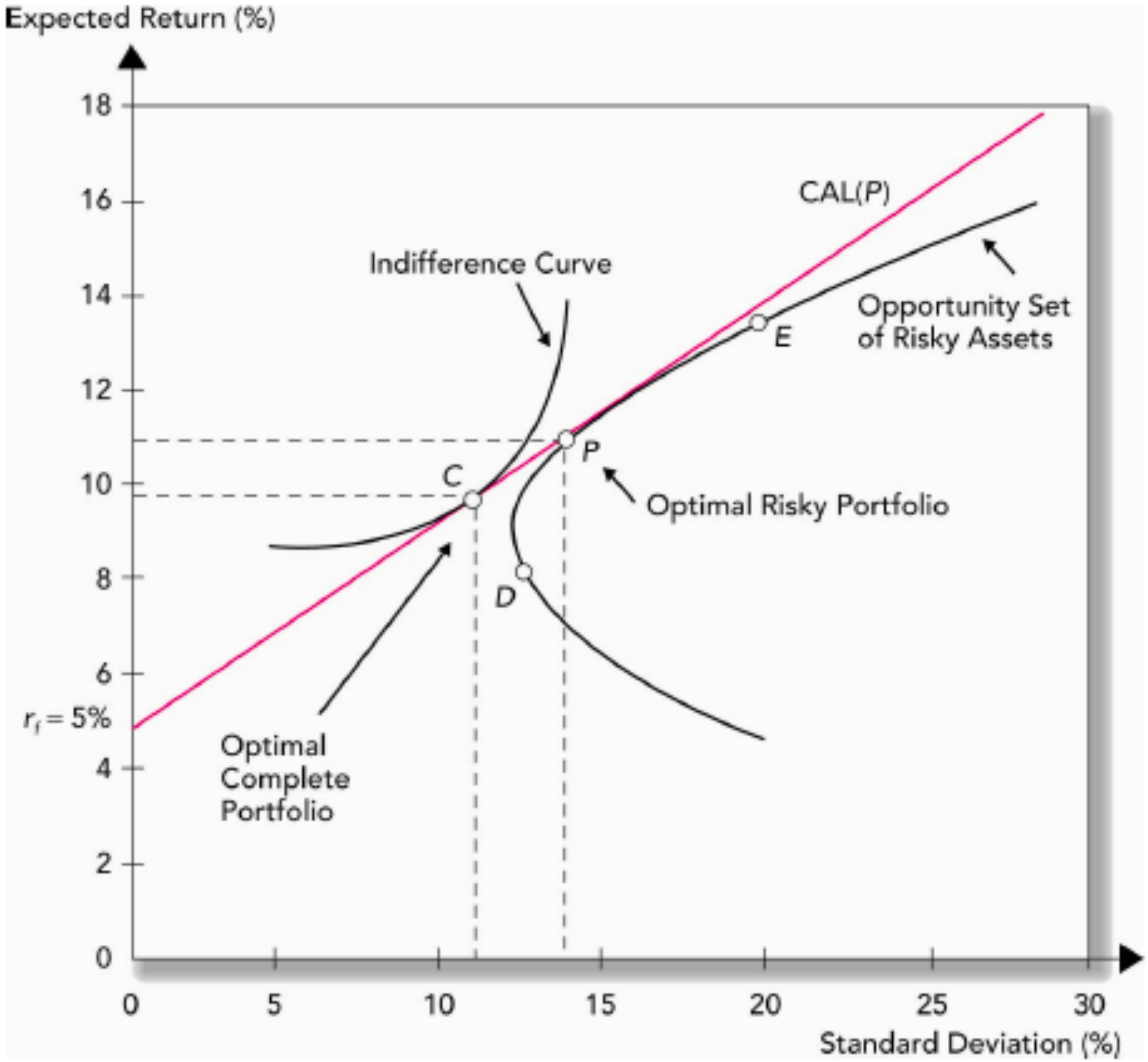
Efficient frontier and tangency portfolio

- Tangency – maximum Sharpe ratio portfolio



Tangency portfolio: optimal combination of risky securities.
Efficient frontier becomes linear: CAL for T

Determination of the Optimal Overall Portfolio



Example

- Two risky assets – bond and stock
  - Means of 3.0% and 7.5%
  - Standard deviations of 10% and 20%
  - Correlation of 0.2
- In this case, the Tangency Portfolio (for r_f=1.5%)
  - Stock weight: 0.6
  - Bond weight: 0.4
  - Expected return: 5.7%
  - Std.dev.: 13.4%,
  - Sharp ratio: 0.31

Example

- Portfolio expected return

  $\mu_p = 0.4 \times 0.03 + 0.6 \times 0.075$

  $= 0.057 = 5.7\%$

- Portfolio variance= $w_x{}^2 \sigma^2{}_x + w_y{}^2 \sigma^2{}_y + 2 w_x w_y \sigma_{xy}$ $\sigma_{xy} = \rho \sigma_x \sigma_y$

  $\sigma_p{}^2 = (0.4)^2 (0.1)^2 + (0.6)^2 (0.2)^2 + 2\ (0.4)(0.6)(0.2 \times 0.1 \times 0.2)$

  $= 0.0180$

- Portfolio standard deviation

  $\sigma_p = \sqrt{0.0180}$

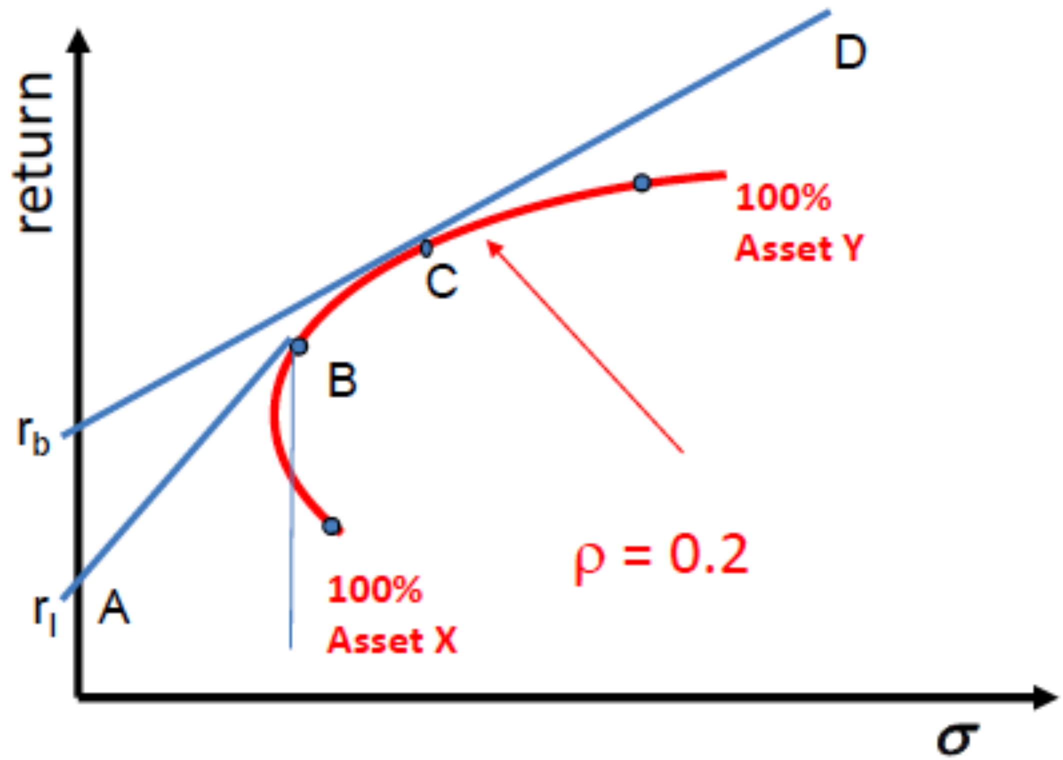
  $= 0.134 = 13.4\%$

  $S = (5.7 - 1.5)/13.4 = 0.31$

Optimal portfolio of tangency and risk-free asset

- If the optimal weight on the tangency portfolio is $w_T = 0.59$
- The weight on the risk-free Tbill is 0.41
- To find the weights on stocks and bonds, multiply the weight on T by the weights that stocks and bonds have in T
  - Weight on stock $0.59 \times 0.6 = 0.35$ (rounding)
  - Weight on bond $0.59 \times 0.4 = 0.24$ (rounding)

Opportunity Set when Borrowing and Lending rates are different



Efficient Portfolios: A-B-C-D

Lessons

- Same risky portfolio (tangency portfolio) chosen by all investors regardless of their risk aversion (separation property).
- Depending on risk aversion, investors choose more or less of the tangency and put the rest in the risk-free asset.
- How do we handle many risky assets?

References

- Berk and DeMarzo, Corporate Finance, 3th edition, 2013.