Question 0:

Consider the function f defined for all (x,y) such that

$$f(x, y; a) = \frac{1}{2}x^2 - x + ay(x-1) - \frac{1}{3}y^3 + a^2y^2,$$

where a is a constant.

a. Prove that $(x^*, y^*) = (1 - a^3, a^2)$ is a stationary point of $f(x, y; a)$.
b. State the condition under which the above stationary point is a global maximum.
c. Given that $G(a) = f(x^*, y^*; a)$, show the derivative of G with respect to a.
d. Calculate $\frac{\partial f(x,y;a)}{\partial a}$ and evaluate its value where $(x^*, y^*) = (1 - a^3, a^2)$. Compare your answer with the answer obtained in b. Are they the same?
e. Determine the domain of (x,y) in the xy-plan where f is convex.

$\frac{\partial f}{\partial x} = x - 1 + ay = 0$

$ax - a + a^2y = 0$ — ①

$\frac{\partial f}{\partial y} = a(x-1) - y^2 + 2a^2y = 0$

$ax - a - y^2 + 2a^2y = 0$ — ②

① − ②

$y^2 = a^2y$

$y = a^2$

plug in y

$x - 1 + a(a^2) = 0$

$x - 1 + a^3 = 0$

$x = 1 - a^3$

$(1-a^3, a^2)$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 1 & a \\ a & 2a^2 - 2a^2 \end{bmatrix}$$

$|H_1| = 1 > 0$

$|H_2| = (0) - (a^2) = -a^2 < 0$

H is negative definite

f is concave at $(1-a^3, a^2)$ → local maximizer

$d^2f < 0 \ \forall x, \forall y$: f is gobal maximum

$(1-a^3, a^2)$ is gobal solution

$\frac{\partial G}{\partial a} = y(x-1) + 2y^2a$

$\therefore y = a^2 \rightarrow 0.76$

$x = 1 - a^3 \rightarrow 0.34$

$a^2(1-a^3-1) + 2(a^2)^2 a = 0$

$2a^5 - a = 0$

$a(2a^5 - 1) = 0$

→ $2a^5 - 1 = 0$

$a = \sqrt[5]{\frac{1}{2}} = 0.87$ $(0.34, 0.76, 0.84)$

$\frac{\partial^2 G}{\partial a^2} = 2y^2 = 2(a^2)^2 = 2a^4$

$$H = \begin{bmatrix} G_{xx} & G_{xy} & G_{xa} \\ G_{yx} & G_{yy} & G_{ya} \\ G_{ax} & G_{ay} & G_{aa} \end{bmatrix} \begin{bmatrix} 1 & a & a^2 \\ a & 0 & 3a^3 \\ a^2 & 3a^3 & 2a^4 \end{bmatrix} \begin{matrix} 1 & a \\ a & 0 \\ a^2 & 3a^3 \end{matrix}$$

$|H_1| = 1 > 0$

$|H_2| = -a^2 < 0$

$|H_3| = (3a^6 + \cancel{3a^6}) - (9a^6 + \cancel{3a^6}) = -6a^6 < 0$

H is negative definite

f is concave at $(0.34, 0.76, 0.84)$ → local maximizer

$d^2f < 0 \ \forall x, \forall y$: f is gobal maximum same as (B)

$(0.34, 0.76, 0.84)$ is gobal solution

e. We need to get $|H_2| > 0$

so a^2 is less than zero : a should be negative

f will be convex

$x - 1 + ay = 0$

$\therefore a < 0 \quad x = 0$

$-1 - ay = 0$

$y = -\frac{1}{a}$ #

Question 2: Suppose that there are three groups of people who take Sky train to commute in Bangkok. The first group is students (s), the second group is senior citizens (old), and the third group is working-aged people. The demand for each group is given by the following equations:

Demand of students: $P_s = 8 - \left(\frac{1}{2}\right) Q_s$

Demand of senior citizens: $P_{old} = 16 - 2Q_{old}$

Demand of working-aged people: $P_w = 20 - Q_w$

The Sky train operator has a constant marginal cost at MC = \$4, and total cost at TC = 4Q + 10. Consider the following problems.

a) Determine the profit-maximizing level of output/price under third-degree price discrimination. Calculate the level of maximized profit.

Max $\pi = TR - TC$

$\pi = (P_s Q_s + P_{old} Q_{old} + P_w Q_w) - [4(Q_s + Q_{old} + Q_w) + 10]$

$\pi = \left((8 - \frac{1}{2}Q_s)Q_s\right) + \left((16 - 2Q_{old})Q_{old}\right) + \left((20 - Q_w)Q_w\right) - [4(Q_s + Q_{old} + Q_w) + 10]$

$\pi = \cancel{8Q_s} - \frac{1}{\cancel{2}}\cancel{Q_s^2} + \cancel{16Q_{old}} - \cancel{2Q_{old}^2} + \cancel{20Q_w} - \cancel{Q_w^2} - \cancel{4Q_s} - \cancel{4Q_{old}} - \cancel{4Q_w} - 10$

$\pi = 4Q_s + 12Q_{old} + 16Q_w - \frac{1}{2}Q_s^2 - 2Q_{old}^2 - Q_w^2 - 10$

FOC.

$\frac{\partial \pi}{\partial Q_s} = 4 - Q_s = 0$

$Q_s = 4$ $\qquad P_s = 8 - \frac{1}{2}(4) = 6$

$\frac{\partial \pi}{\partial Q_{old}} = 12 - 4Q_{old} = 0$

$Q_{old} = 3$ $\qquad P_{old} = 16 - 2(3) = 10$

$\frac{\partial \pi}{\partial Q_w} = 16 - 2Q_w = 0$

$Q_w = 8$ $\qquad P_w = 20 - 8 = 12$

$\pi = (P_s Q_s + P_{old} Q_{old} + P_w Q_w) - [4(Q_s + Q_{old} + Q_w) + 10]$

$\pi = (6(4) + 10(3) + 12(8)) - (4(4+3+8) + 10)$

$\pi = (24 + 30 + 96) - 70$

$\pi = 80$ #

b) Confirm your result in (a) with the second-order derivative test.

SOC.

$$H = \begin{bmatrix} \pi_{ss} & \pi_{sold} & \pi_{sw} \\ \pi_{olds} & \pi_{oldold} & \pi_{oldw} \\ \pi_{ws} & \pi_{wold} & \pi_{ww} \end{bmatrix}$$

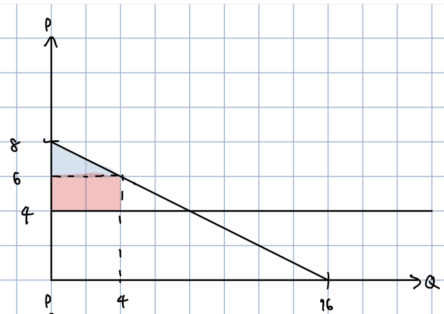
$$H = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$|H_1| = -1 < 0$

$|H_2| = \begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix} = 4 > 0$

So H is negative definite. Hence, $Q_s = 4$, $Q_{old} = 3$, $Q_w = 8$, $P_s = 6$, $P_{old} = 10$, $P_w = 12$ that is maximize profit

c) Calculate (i) consumer surplus for each of the three groups of consumers, and (ii) producer surplus.



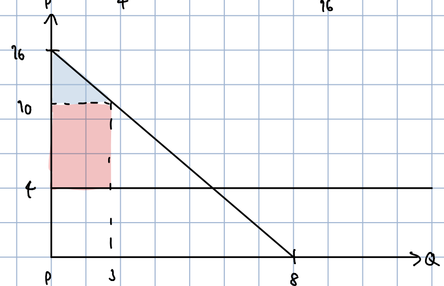
Students

consumer surplus $= \frac{1}{2} \times (8-6) \times 4$

$= 4$

producer surplus $= (6-4)(4)$

$= 8$



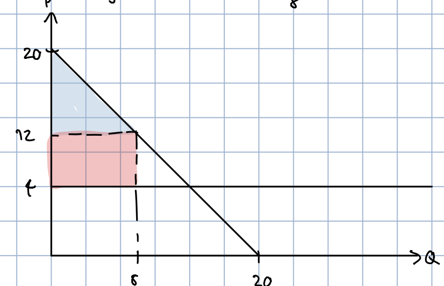
Senior

consumer surplus $= \frac{1}{2} \times (16-10) \times 3$

$= 9$

producer surplus $= (10-4)(3)$

$= 18$



Working-aged

consumer surplus $= \frac{1}{2} \times (20-12) \times \overset{4}{\cancel{8}}$

$= 32$

producer surplus $= (12-4) \times 8$

$= 64$

d) Calculate the optimal level of total output if the Sky train operator can practice the first-degree price discrimination for each group of the consumers in the market.

Question 4:

Suppose that the output Q of a firm depends on two inputs of the quantities K and L. The output level is determined by the production function

$$Q = 36K + 16L - 3K^2 - 2KL - L^2$$

a. Is the firm's production function strictly concave? Explain.

$\frac{\partial Q}{\partial K} = 36 - 6K - 2L$

$\frac{\partial Q}{\partial L} = 16 - 2K - 2L$

check that $d^2Q < 0$

$$H = \begin{bmatrix} Q_{KK} & Q_{KL} \\ Q_{LK} & Q_{LL} \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ -2 & -2 \end{bmatrix}$$

$|H_1| = -6 < 0$

$|H_2| = (-6)(-2) - (-2)^2$

$= 12 - 4 = 8 > 0$

$\therefore$ As $|H_1| < 0$ and $|H_2| > 0$ with alternate sign for $\forall K, L$, we can conclude that d^2Q is negative definite. Therefore, the function is globally concave

b. Determine the optimal input (K^*, L^*) that maximizes the output level.

$\frac{\partial Q}{\partial K} = 36 - 6K - 2L = 0$

$2L + 6K = 36$ ①

$\frac{\partial Q}{\partial L} = 16 - 2K - 2L = 0$

$2L + 2K = 16$ ②

① − ② $= \cancel{2L} + 6K - \cancel{2L} - 2K = 36 - 16$

$4K = 20$ $\quad K^* = 5$

② $2L + 2(5) = 16$

$2L = 6$

$L^* = 3$

$\therefore K^* = 5, L^* = 3$

c. Write down the firm's profit function when the price of Q is P and the per-unit factor prices of K and L are r and w, respectively, where both r and w are positive numbers. Find the levels of K^* and L^* that maximize the firm's profits.

$\pi = P \cdot (36K + 16L - 3K^2 - 2KL - L^2) - rK - wL$

$\frac{\partial \pi}{\partial K} = 36P - 6PK - 2PL - r = 0$

$6PK = 36P - 2PL - r$

$K = 6 - \frac{1}{3}L - \frac{r}{6P}$

$K^* = 6 - \frac{1}{3}\left(3 + \frac{r}{4P} - \frac{3w}{2}\right) - \frac{r}{6P}$

$= 5 - \frac{r}{12P} + \frac{w}{2} - \frac{r}{6P}$

$K^* = 5 + \frac{w}{2} - \frac{\cancel{3}r}{\cancel{12}P} = 5 + \frac{w}{2} - \frac{r}{4P}$

$\frac{\partial \pi}{\partial L} = 16P - 2PL - 2PK - w = 0$

$2PL = 16P - 2PK - w$

$L = 8 - K - w$

$L = 8 - \left(6 - \frac{1}{3}L - \frac{r}{6P}\right) - w$

$= 2 + \frac{1}{3}L + \frac{r}{6P} - w$

$\frac{2}{3}L = 2 + \frac{r}{6P} - w$

$L^* = 3 + \frac{r}{4P} - \frac{3w}{2}$

d. Verify that the second-order sufficient conditions for maximum profits are satisfied.

e. Determine the effect of an increase in r on the firm's use of each input. (i.e. determine $\frac{\partial K^*}{\partial r}$ and $\frac{\partial L^*}{\partial r}$).

D. $\frac{\partial \pi}{\partial K} = 36P - 6PK - 2PL - r$, $\frac{\partial \pi}{\partial L} = 16P - 2PL - 2PK - w = 0$

S.O.C. $\begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix}$

$\frac{\partial^2 \pi}{\partial K^2} = -6P$, $\frac{\partial^2 \pi}{\partial K \partial L} = -2P$, $\frac{\partial^2 \pi}{\partial L \partial K} = -2P$, $\frac{\partial^2 \pi}{\partial L^2} = -2P$

$\begin{bmatrix} -6P & -2P \\ -2P & -2P \end{bmatrix} \rightarrow$ $|H_1| = -6P < 0$

$|H_2| = (-6P)(-2P) - (-2P)(-2P) = 8P^2 > 0$ *

Hence, H is negative definite $|H_1| < 0$, $|H_2| > 0$; $\partial^2 \pi = 0$ at K^*, L^* is local maximizer.

So π is globally concave.

e. $K^* = 5 + \frac{w}{2} - \frac{r}{4P}$, $L^* = 3 + \frac{r}{4P} - \frac{3w}{2}$

$\frac{\partial K^*}{\partial r} = -\frac{1}{4P}$ * , $\frac{\partial L^*}{\partial r} = \frac{1}{4P}$ *

When K^* increase by 1 unit, r will decrease $-\frac{1}{4P}$. But when L^* increase by 1 unit r will increase by $\frac{1}{4P}$.