

Optimization with Equality Constraints V

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CW Ch.12

Outline

- Least-Cost Combination of Inputs

Least-Cost Combination of Inputs: First-Order Condition

- Assuming a smooth production function with two inputs $Q = Q(a, b)$, where $Q_a, Q_b > 0$, and assuming both input prices to be exogenous, the cost minimization problem can be expressed as

$$\begin{aligned} \min_{a,b} C &= aP_a + bP_b \\ \text{s.t} \quad Q(a, b) &= Q_0. \end{aligned}$$

- The Lagrangian equation is

$$Z = aP_a + bP_b + \mu[Q_0 - Q(a, b)].$$

- **FOC:**

$$Z_a = P_a - \mu Q_a = 0$$

$$Z_b = P_b - \mu Q_b = 0$$

$$Z_\mu = Q_0 - Q(a, b) = 0$$

Least-Cost Combination of Inputs: First-Order Condition

- The first two equations imply the condition

$$\frac{P_a}{Q_a} = \frac{P_b}{Q_b} = \mu.$$

- At the optimal input combination, the input-price-marginal-product ratio must be the same for each input.
- Since this ratio measures the amount of expenditure per unit of marginal product of the input under consideration, the Lagrange multiplier can be given the interpretation of the marginal cost of production in the optimum state. Thus,

$$\mu^* = \frac{\partial C^*}{\partial Q_0}.$$

Least-Cost Combination of Inputs: First-Order Condition

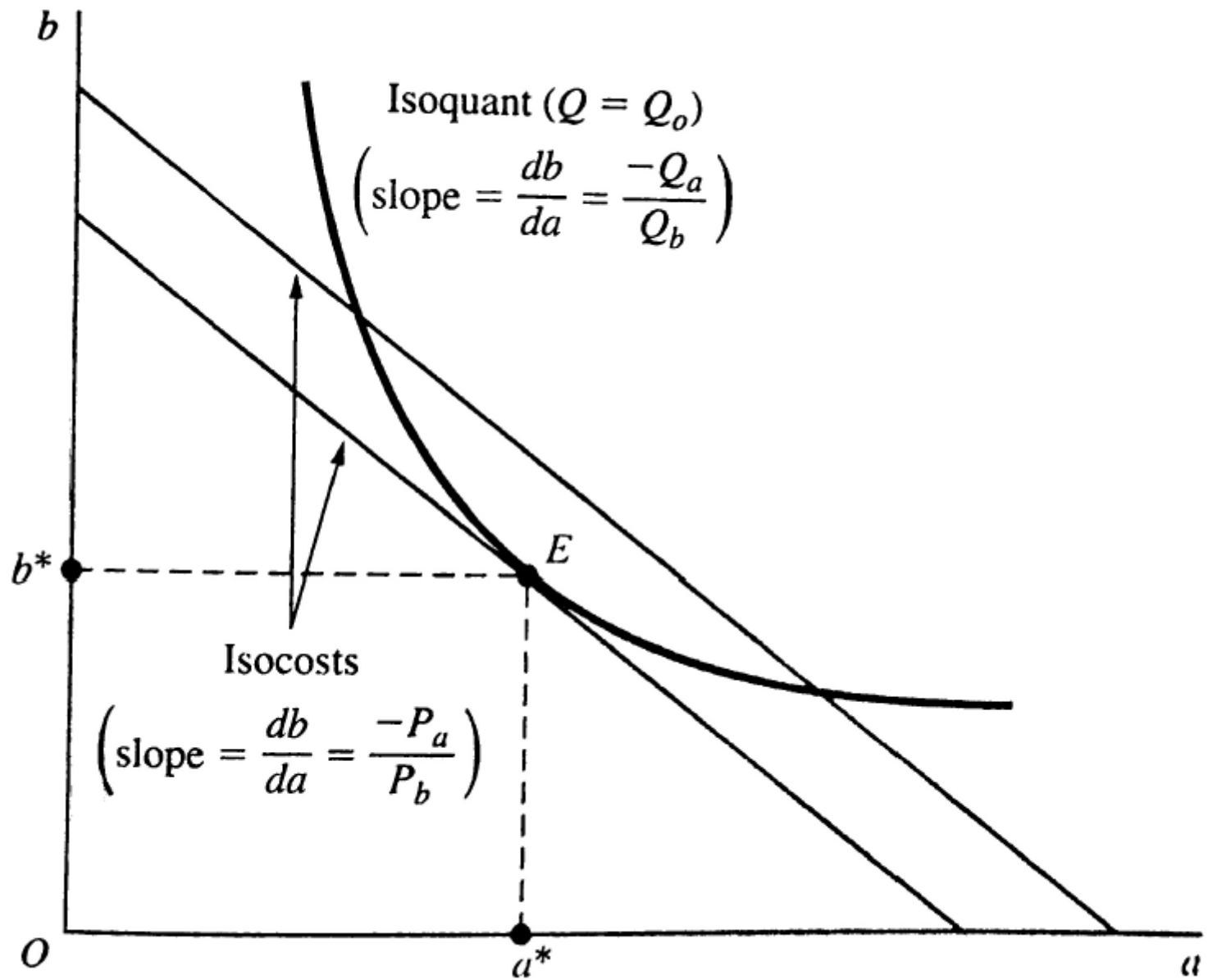
- Alternatively, the optimal condition can be expressed as:

$$\frac{P_a}{P_b} = \frac{Q_a}{Q_b},$$

which reflects the point where the slope of the isoquant equal to the slope of the isocost.

- The slope of an isoquant is usually called the *marginal rate of technical substitution of a for b* ($MRTS_{ab}$), which measures the rate of exchange of inputs a and b that can keep the production constant.
- The slope of an isocost can be seen from

$$C_0 = aP_a + bP_b \rightarrow b = \frac{C_0}{P_b} + \frac{P_a}{P_b} a.$$



Least-Cost Combination of Inputs: Second-Order Condition

- To ensure a minimum cost, it is sufficient (after the FOC is met) to have a negative bordered Hessian:

$$|\bar{H}| = \begin{vmatrix} 0 & Q_a & Q_b \\ Q_a & -\mu Q_{aa} & -\mu Q_{ab} \\ Q_b & -\mu Q_{ba} & -\mu Q_{bb} \end{vmatrix} = \mu(Q_{aa}Q_b^2 - 2Q_{ab}Q_aQ_b + Q_{bb}Q_a^2) < 0.$$

- Since the optimal value of μ (marginal cost) is positive, it reduces to the condition in parentheses be negative at the optimal point.
- It can be shown that the second derivative of an isoquant is

$$\frac{d^2b}{da^2} = \frac{-1}{Q_b^3} (Q_{aa}Q_b^2 - 2Q_{ab}Q_aQ_b + Q_{bb}Q_a^2).$$

Satisfaction of the SOC imply that $\frac{d^2b}{da^2}$ is positive. That is the isoquant is strictly convex.

- Note also that this SOC is just sufficient, not necessary.

Least-Cost Combination of Inputs: The Expansion Path

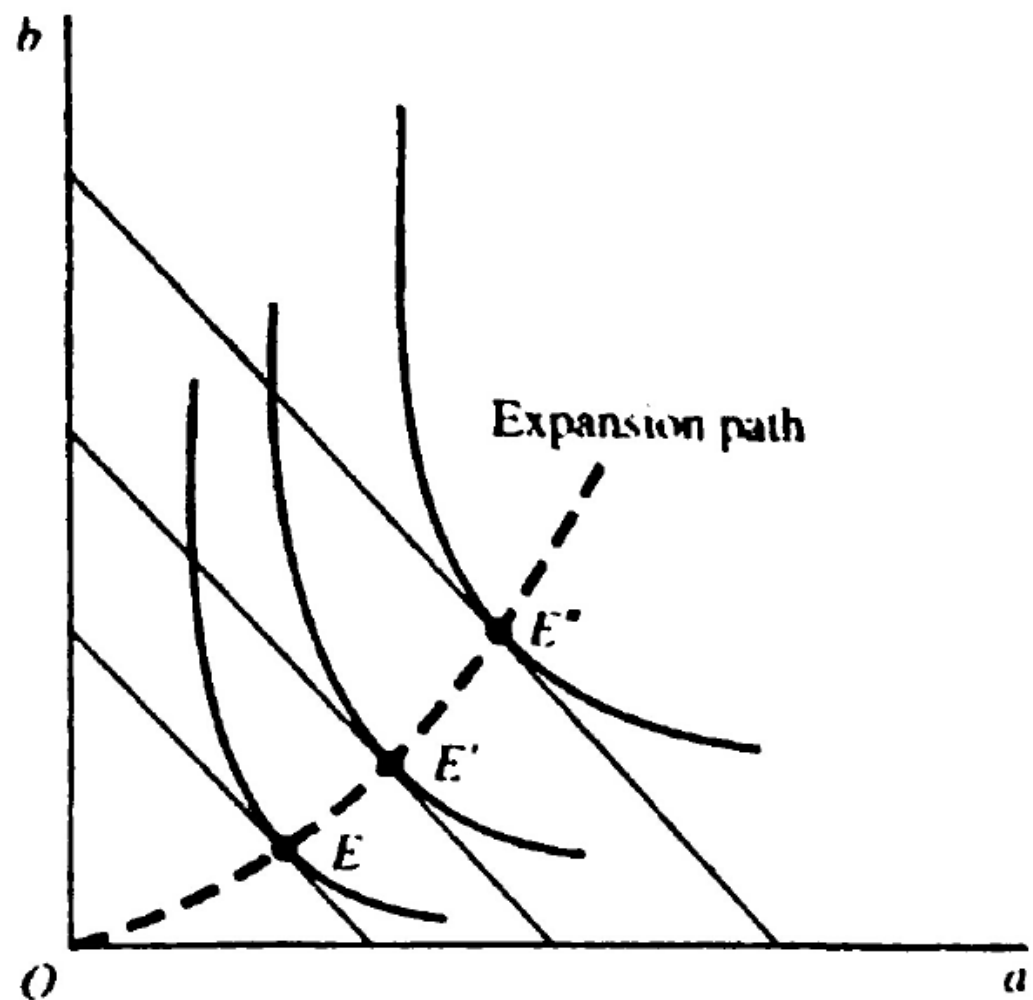
- Now, let us postulate successive increases of Q_0 and trace the effect on the least-cost combination $\frac{b^*}{a^*}$.
- The locus of such points, known as the **expansion path** of the firm, serves to describe the least-cost combinations required to produce varying levels of Q_0 .
- If we assume the strict convexity of the isoquants, the expansion path will be derivable directly from the first-order condition.
- For the function $Q = Aa^\alpha b^\beta$, the FOC implies that each point on the expansion path must satisfy

$$\frac{P_a}{P_b} = \frac{Q_a}{Q_b} = \frac{\alpha A a^{\alpha-1} b^\beta}{\beta A a^\alpha b^{\beta-1}} = \frac{\alpha b}{\beta a},$$

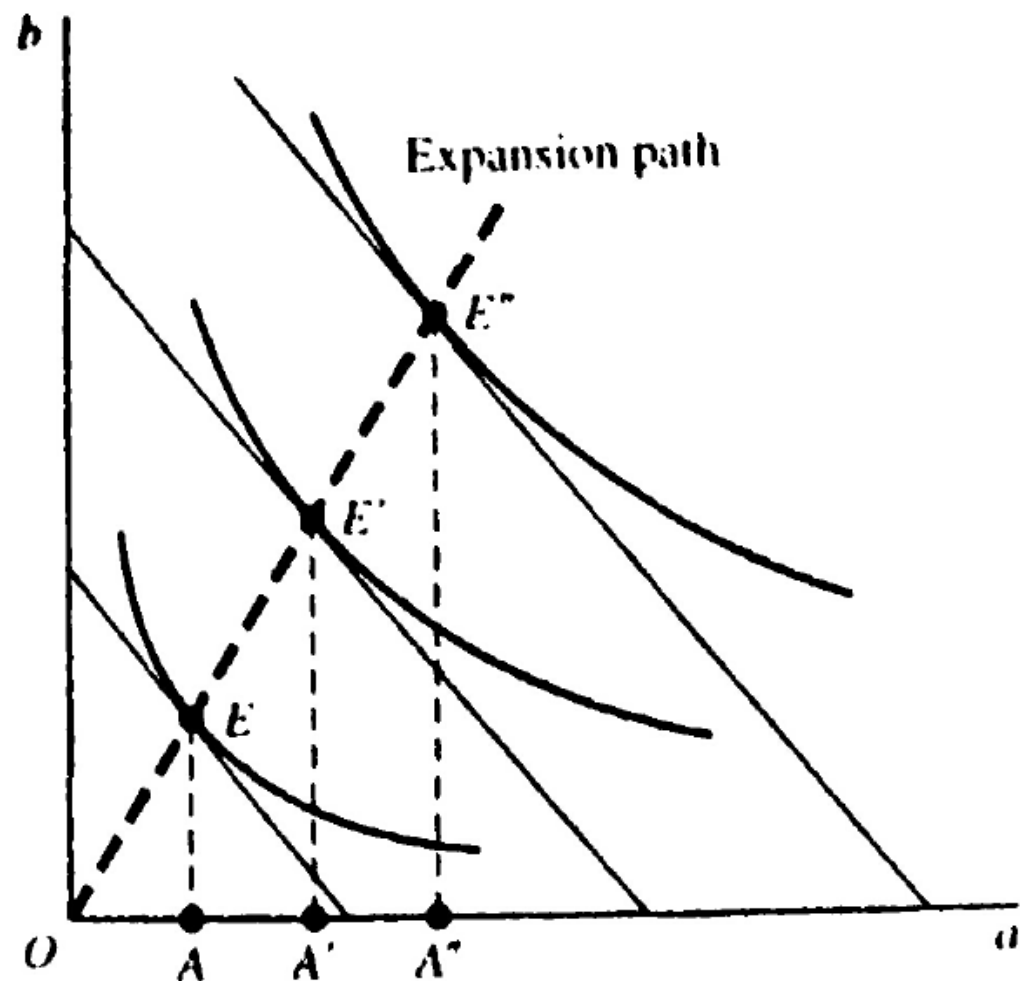
Implying that the optimal input ratio should be

$$\frac{b^*}{a^*} = \frac{\beta P_a}{\alpha P_b} \text{ (a constant).}$$

All points on the expansion path must show the same *fixed* input ratio.



(a)



(b)

Least-Cost Combination of Inputs: The Expansion Path

- The linearity of the expansion path is characteristic of the generalized Cobb-Douglas function whether or not $\alpha + \beta = 1$.
- Actually, any homogeneous production function (not necessarily the Cobb-Douglas) will give rise to a linear expansion path for each set of input prices.
- Although any homogeneous production function yields a linear expansion path, the specific degree of homogeneity does make a significant difference to the amount of production on the points along the path. That is the spacing of the isoquants for $Q = 1, Q = 2, \dots$, will be widely different for different degrees of homogeneity.

Least-Cost Combination of Inputs: Elasticity of Substitution

- Another aspect of comparative statics has to do with the effect of a change in the P_a/P_b ratio upon the least-cost combination b^*/a^* .
- When P_a/P_b rises, a becomes relatively more expensive, we can expect b^*/a^* to rise as well. But what is the extent of the rise?
- One way to measure the extent of the rise is to use the elasticity of substitution, denoted by σ , where

$$\sigma \equiv \frac{\text{relative change in } \left(\frac{b^*}{a^*}\right)}{\text{relative change in } \left(\frac{P_a}{P_b}\right)} = \frac{\frac{d\left(\frac{b^*}{a^*}\right)}{\frac{b^*}{a^*}}}{\frac{d\left(\frac{P_a}{P_b}\right)}{\frac{P_a}{P_b}}} = \frac{d\left(\frac{b^*}{a^*}\right)}{\frac{b^*}{a^*} \frac{d\left(\frac{P_a}{P_b}\right)}{\frac{P_a}{P_b}}}.$$

Least-Cost Combination of Inputs: Elasticity of Substitution

- For the function $Q = Aa^\alpha b^\beta$, previously we have

$$\frac{b^*}{a^*} = \frac{\beta P_a}{\alpha P_b} = \frac{\beta}{\alpha} \left(\frac{P_a}{P_b} \right),$$

which yields

$$\frac{d\left(\frac{b^*}{a^*}\right)}{d\left(\frac{P_a}{P_b}\right)} = \frac{\beta}{\alpha}, \text{ and } \frac{\frac{b^*}{a^*}}{\frac{P_a}{P_b}} = \frac{\beta}{\alpha}.$$

- Substituting into the formula of the previous page, we find that $\sigma = 1$.
- For the generalized Cobb-Douglas production function, the elasticity of substitution is a constant, equal to one. If $\frac{P_a}{P_b}$ increases by one percent, $\frac{b^*}{a^*}$ increases by one percent as well.

Least-Cost Combination of Inputs: CES Production Function

- More recently, another form of production function is also widely used. This is the ***constant elasticity of substitution (CES)*** production function.
- A CES production function yields constant σ need not be one. The function can be written as

$$Q = A[\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-1/\rho}, \quad 0 < \delta < 1, -1 < \rho \neq 0.$$

- This function is homogeneous of degree one:

$$\begin{aligned} A[\delta(jK)^{-\rho} + (1 - \delta)(jL)^{-\rho}]^{-1/\rho} &= A\{j^{-\rho}[\delta K^{-\rho} + (1 - \delta)L^{-\rho}]\}^{-1/\rho} \\ &= Aj^{\rho/\rho}[\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-1/\rho} \\ &= jQ. \end{aligned}$$

Hence, it displays constant returns to scale.

Least-Cost Combination of Inputs: CES Production Function

- The isoquants of CES production function are always negatively sloped and strictly convex for positive values of K and L . To show this, first we figure out Q_L and Q_K

$$\begin{aligned}Q_L &\equiv \frac{\partial Q}{\partial L} = A \left(-\frac{1}{\rho} \right) [\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-\frac{1}{\rho}-1} (1 - \delta)(-\rho)L^{-\rho-1} \\&= (1 - \delta)A[\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-\frac{1+\rho}{\rho}} L^{-(1+\rho)} \\&= (1 - \delta) \frac{A^{1+\rho}}{A^\rho} [\delta K^{-\rho} + (1 - \delta)L^{-\rho}]^{-\frac{1+\rho}{\rho}} L^{-(1+\rho)} \\&= \frac{(1-\delta)}{A^\rho} \left(\frac{Q}{L} \right)^{1+\rho} .\end{aligned}$$

Similarly,

$$Q_K \equiv \frac{\partial Q}{\partial K} = \frac{\delta}{A^\rho} \left(\frac{Q}{K} \right)^{1+\rho} .$$

Least-Cost Combination of Inputs: CES Production Function

- Next, from the implicit-function rule,

$$\frac{dK}{dL} = -\frac{Q_L}{Q_K} = -\frac{1-\delta}{\delta} \left(\frac{K}{L}\right)^{1+\rho} < 0.$$

From this, it can be shown that $\frac{d^2K}{dL^2} > 0$.

- At last, from the optimal condition $\frac{Q_L}{Q_K} = \frac{P_L}{P_K}$, we have

$$\frac{1-\delta}{\delta} \left(\frac{K}{L}\right)^{1+\rho} = \frac{P_L}{P_K} \rightarrow \left(\frac{K^*}{L^*}\right) = \left(\frac{\delta}{1-\delta}\right)^{\frac{1}{1+\rho}} \left(\frac{P_L}{P_K}\right)^{\frac{1}{1+\rho}}.$$

From this we can find that

$$\sigma = \frac{1}{1+\rho} \text{ (a constant).}$$

Least-Cost Combination of Inputs: CES Production Function

- The value of σ depends on the value of ρ as follows

$$\left. \begin{array}{l} -1 < \rho < 0 \\ \rho = 0 \\ 0 < \rho \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \sigma > 1 \\ \sigma = 1 \\ \sigma < 1 \end{array} \right.$$

- Note here that Cobb-Douglas function is a special case of the CES function. As $\rho \rightarrow 0$, the CES function approaches the Cobb-Douglas function.