

Assignment 3: Due date: March 31, 2022 before 2.00 pm

On page 134-138, Chapter 4: Consumption-savings and state pricing, work on the following questions:

2, 3, 4, 6

2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^k \pi_s m_s X_{sa} \\ &= E[mX_a] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E[\delta \ln(C_1)]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s , $s = 1, \dots, k$. Denote the date 0 price of elementary security s as p_s , and derive an expression for it in terms of the individual's equilibrium consumption.

$$p_s = \pi_s m_s$$

$$\text{and } m_s = \frac{\delta v'(C_s^*)}{v'(C_0^*)} = \frac{\delta C_0^*}{C_s^*}$$

$$\therefore p_s = \pi_s \delta \frac{C_0^*}{C_s^*}$$

3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E[m_{01}R_s]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s . Also assume markets are complete and consider the above relationship for primitive security s ; that is, let R_s be the rate of return on primitive (or elementary) security s . The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d(C_s/C_0)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s . If the individual's expected utility is given by

$$U(C_0) + \delta E[U(\tilde{C}_1)]$$

where utility displays constant relative risk aversion, $U(C) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I .

$$m_{01} = \frac{\delta U'(C_1)}{U'(C_0)} = \delta \left(\frac{C_1}{C_0}\right)^{\gamma-1}$$

$$\text{FOC: } \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} R_s = 1 \quad (1) \quad \text{Then differentiate}$$

$$\pi_s \delta (\gamma-1) \left(\frac{C_s}{C_0}\right)^{\gamma-2} R_s d\left(\frac{C_s}{C_0}\right) + \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} dR_s = 0$$

$$(\gamma-1) \left(\frac{C_s}{C_0}\right)^{\gamma-1} R_s d\left(\frac{C_s}{C_0}\right) = -dR_s$$

$$\frac{R_s}{C_s/C_0} \frac{d(C_s/C_0)}{dR_s} = \frac{-dR_s}{(\gamma-1)}$$

$$\frac{R_s}{C_s/C_0} \frac{d(C_s/C_0)}{dR_s} = \frac{-1}{\gamma-1}$$

$$\frac{R_s}{(C_s/C_0)} \frac{d(C_s/C_0)}{dR_s} = \frac{1}{(1-\gamma)}$$

4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.¹⁶ There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

- a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

$$\text{Let } P = \begin{bmatrix} 1/1.05 \\ 6 \end{bmatrix} \text{ and } X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}$$

$$\begin{aligned} \text{Then } [p_1 \ p_2] &= P'X^{-1} \\ &= \begin{bmatrix} 1/1.05 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix} \\ &= [0.2476 \quad 0.7048] \end{aligned}$$

For the risk neutral probabilities:

$$\hat{\pi}_1 = p_1 R_f = 0.2476 \times 1.05 = 0.26$$

$$\hat{\pi}_2 = p_2 R_f = 0.7048 \times 1.05 = 0.74$$

- b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$.

What is the stochastic discount factor for the two states?

$$m_1 = \frac{p_1}{\pi_1} = \frac{0.2476}{0.5} = 0.4952 = 0.495$$

$$m_2 = \frac{p_2}{\pi_2} = \frac{0.7048}{0.5} = 1.4096 = 1.41$$

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i , R_f is the risk-free return, β_i is asset i 's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2 .

- Derive an expression for γ as a function of a , b , $E[R_m]$, and σ_m^2 . (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)
- Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f , $E[R_m]$, and σ_m^2 .
- Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

$$\begin{aligned}
 \text{a)} \quad 0 &= E[m(R_i - R_f)] \\
 &= E[(a + bR_m)(R_i - R_f)] \\
 &= aE[R_i] - aR_f + bE[R_m R_i] - bR_f E[R_m] \\
 &= a(E[R_i] - R_f) + b(E[R_m]E[R_i] + \text{Cov}[R_m, R_i] - R_f E[R_m]) \\
 &= (E[R_i] - R_f)(a + bE[R_m]) + b \text{Cov}[R_m, R_i]
 \end{aligned}$$

$$\begin{aligned}
 E[R_i] - R_f &= \frac{-b \text{Cov}[R_m, R_i]}{a + bE[R_m]} \\
 &= \frac{-\text{Cov}[R_m, R_i]}{\sigma_m^2} \frac{b \sigma_m^2}{a + bE[R_m]} \\
 &= -\beta_i \frac{b \sigma_m^2}{a + bE[R_m]}
 \end{aligned}$$

$$r = \frac{-b \sigma_m^2}{a + bE[R_m]}$$

$$\begin{aligned} \text{b) Risk free asset : } \frac{1}{R_f} &= E[a + bR_m] \\ a &= \frac{1}{R_f} - bE[R_m] \end{aligned}$$

market portfolio :

$$\begin{aligned} 1 &= E[(a + bR_m)R_m] \\ &= aE[R_m] + bE[R_m^2] \\ &= aE[R_m] + b(\sigma_m^2 + E[R_m]^2) \quad \text{---(1) substitute a into (1)} \end{aligned}$$

$$1 = \left[\frac{1}{R_f} - bE[R_m] \right] E[R_m] + b(\sigma_m^2 + E[R_m]^2)$$

$$1 = \frac{E[R_m]}{R_f} + b\sigma_m^2$$

find "b"

$$1 - \frac{E[R_m]}{R_f} = b\sigma_m^2$$

$$\frac{R_f - E[R_m]}{R_f} = b\sigma_m^2$$

$$\frac{R_f - E[R_m]}{R_f\sigma_m^2} = b$$

find "a"

$$a = \frac{1}{R_f} - bE[R_m]$$

$$= \frac{1}{R_f} \times \frac{\sigma_m^2}{\sigma_m^2} - \left[\frac{R_f - E[R_m]}{R_f\sigma_m^2} \right] E[R_m]$$

$$= \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f\sigma_m^2}$$

$$\text{c) } a + bE[R_m] = \frac{\sigma_m^2 + E[R_m](E[R_m] - R_f)}{R_f\sigma_m^2} + E[R_m] \left[\frac{R_f - E[R_m]}{R_f\sigma_m^2} \right]$$

$$= \frac{\sigma_m^2}{R_f\sigma_m^2}$$

$$= \frac{1}{R_f}$$

$$\therefore r = \frac{-b\sigma_m^2}{a + bE[R_m]}$$

$$= - \left(\frac{R_f - E[R_m]}{R_f\sigma_m^2} \right) \sigma_m^2 R_f = E[R_m] - R_f$$