Minimum Mean-Square-Error (MSE) Estimator

The MSE of an estimator $E[E(\hat{\theta})-\theta]^2 = [E(\hat{\theta})-\theta]^2$ $\hat{\theta}$ is defined as

$$MSE(\hat{\theta}) = E(\hat{\theta}-\theta)^2$$
$$Var(\hat{\theta}) = E[\hat{\theta}-E(\hat{\theta})]^2$$

The difference between the two is that $Var(\hat{\theta})$ measures the dispersion of the distribution of $\hat{\theta}$ around its mean or expected value, whereas $MSE(\hat{\theta})$ measures dispersion around the true value of the parameter

$$\begin{aligned} MSE(\hat{\theta}) &= E(\hat{\theta}-\theta)^2 \\ &= E(\hat{\theta}-E(\hat{\theta})+E(\hat{\theta})-\theta)^2 \\ &= E[\hat{\theta}-E(\hat{\theta})]^2 + E[E(\hat{\theta})-\theta]^2 + 2E[\hat{\theta}-E(\hat{\theta})][E(\hat{\theta})-\theta] \\ &= Var(\hat{\theta}) + bias(\hat{\theta})^2 \end{aligned}$$

$$2E[\hat{\theta}-E(\hat{\theta})][E(\hat{\theta})-\theta]$$
$$= 2\{[E(\hat{\theta})]^2 - [E(\hat{\theta})]^2 - \theta E(\hat{\theta}) + \theta E(\hat{\theta})\} = 0$$

$E[E(\hat{\theta})-\theta]^2 = [E(\hat{\theta})-\theta]^2$ Since the expected value of a constant is simply the constant itself.

If the bias is zero $MSE(\hat{\theta}) = Var(\hat{\theta})$