Tutorial 1 – Multivariable Optimasation

Multi-variable Unconstraint Optimisation

1. $f(x,y) = 5x^3 + 6x^2y^2 + 3xy^3 - 2y + 1$ Find $f(0,1)$, $f(1,0)$, $f(a,2a)$

[Ans: -1, 6, $f(a,2a) = 48a^4 + 5a^3 - 4a + 1$]

2. Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^3 f}{\partial x^2 \partial y}$ of the following equations

2.1. $f(x,y) = 5x^3 + 6x^2y^2 + 3xy^3 - 2y + 1$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 24y$]

2.2. $f(x,y) = 2(x + y^2)^3 + 5xy + 4$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 24y$]

2.3. $f(x,y) = \frac{5}{x-y} + \frac{2}{y} + xy^3$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{30}{(x-y)^4}$]

2.4. $f(x,y) = (2x + 3y)\, e^{4x+5y}$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 16e^{4x+5y}(8 + 10x + 15y)$]

2.5. $f(x,y) = \ln(5xy) - e^{2x} + 1/y$ [Ans : $\frac{\partial^3 f}{\partial x^2 \partial y} = 0$]

3. Find $\frac{\partial z}{\partial x}$ where $z = f(x,y)$ from

3.1. $xz^2 = 5x^2z + 3xy - 4$ [Ans: $\frac{\partial z}{\partial x} = \frac{10xz + 3y - z^2}{2xz - 5x^2}$]

3.2. $2xy = 4e^{3xz} + 2yz$ [Ans: $\frac{\partial z}{\partial x} = \frac{y - 6ze^{3xz}}{y + 6xe^{3xz}}$]

4.

4.1 $z = f(x,y) = xy^2 + 3x^2y - 4y + 1$, $x = g(r,s) = r - s^3 + 5$, $y = h(r,s) = 2rs$. Find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial s}$, given $r = 1$, $s = 1$ [Ans: 246, -10]

4.2 $z = f(x,y) = 2x + x^2y^2 - \frac{1}{xy}$, $x = g(u,v,w) = 2u - vw$, $y = h(u,v) = -uv^2$. Find $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}, \frac{\partial z}{\partial w}$.

[Ans:

$\frac{\partial z}{\partial u} = 2\left(2 + 2xy^2 + \frac{1}{x^2y}\right) - v^2\left(2x^2y - \frac{1}{xy^2}\right), \frac{\partial z}{\partial v} = w\left(2 + 2xy^2 + \frac{1}{x^2y}\right) - 2uv\left(2x^2y - \frac{1}{xy^2}\right), \frac{\partial z}{\partial w} = -v\left(2 + 2xy^2 + \frac{1}{x^2y}\right)$]

5. Find all the critical points and classify whether they are relative maximum, relative minimum, or saddle point.

5.1. $f(x,y) = -x^2 + 4x - 4y^2 + 8y + 1$ [Ans : max at (2,1,9)]

5.2. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$ [Ans: (-1,2) (-1,-2) saddle, (0,0) min, (-5/3,0) max]

6. A firm seeks to maximise profits. Using the inputs K and L (capital and labour), it produces X units of product, based on the production function

$$P = 100K^{0.25}L^{0.25} - 2K - 2L$$

Find the values of K and L which give the firm maximum profit. How much is the maximum profit? [Ans: 156.25,156.25,625]

7. Find all relative maximum and minimum points of the functions:

(i) $Y = 6 - X^2 - Z^2$

(ii) $Y = (X-2)^2 + (Z-4)^4$

(iii) $Y = (X-2)^2 - (Z-4)^4$

(iv) $Y = (X-1)^2 + (Z-1)^2 + (X-1)^2(Z-1)^2$

[Ans: (i) x=0, z=0 maximum, (ii) x=2, z=4 neither, (iii) x=2, z=4, neither (iv) x=1, z=1, minimum]

8. A firm can sell its product in two countries, A and B, where demand in country A is given by P_A=100-2Q_A and in country B is given by P_B=100-Q_B. Its total output is Q_A+Q_B which it can produce at a cost of $50(Q_A + Q_B) + \frac{1}{2}(Q_A + Q_B)^2$. Show that its total profit is given by:

$$100Q_A - 2Q_A^{\ 2} + 100Q_B - Q_B^2 - 50(Q_A + Q_B) - \tfrac{1}{2}(Q_A + Q_B)^2$$

How much will it sell in the two countries assuming it maximises profits?

[Ans: $Q_A = 7\frac{1}{7} \quad Q_B = 14\frac{2}{7}$]

9. If a firm spends b per unit of output on advertising then it can sell q units of output at a price given by

$$p = 100 - 2q + 8b^{1/2}$$

In order to produce q units of output it costs 4q+q^2. Show that its profit π is given by

$$\pi = 100q - 2q^2 + 8qb^{1/2} - 4q - q^2 - bq\ .$$

If the firm maximises profits, how much will it spend on advertising and how much will it produce?

[Ans: $q = 18\frac{2}{3} \quad b = 16$]

Multi-variable Constraint Optimisation

10. Use **the substitution method**, solve the following constrained optimisation problem. Determine the optimum values of the choice variables, and use the second-order condition to verify that they are relative minima or maxima.

(a) $y = x^2 + 2xz + 4z^2$ subject to $x + z = 8$ [Ans: $(x, z) = (8,0)$ relative minimum]

(b) $y = 10x + 40z$ subject to $x^{\frac{1}{2}} z^{\frac{1}{2}} = 2$ [Ans: $(x, z) = (4,1)$ relative minimum]

(c) $y = \ln 2x + 2\ln z$ subject to $z = 16 - 4x$ [Ans: $(x, z) = \left(\frac{4}{3}, \frac{32}{3}\right)$ relative maximum]

11. Use **the substitution method** to find the optimum points for the multivariable function

$$U = 2x^2 + 5xy - y^2 - 3xz$$

subject to

$$x + y + z = 14.$$

Use the second-order derivative test to confirm whether the extreme value represents a relative maximum, relative minimum or saddle point. Ans: $(x, y, z) = (1,4,9)$ saddle point]

12. (a) Find the optimum values of

$$f(x, y) = 3x^2 + xy + 2y^2$$

subject to the constraint

$$x + y = 200.$$

[Ans: $f(75,125) = 57{,}500$]

(b) A firm producing 2 products, Product *X* with quantity *x* and Product *Y* with quantity *y*. The total cost is

$$C = 3x^2 + xy + 2y^2.$$

The capacity of production is

$$x + y \le 200.$$

Find the optimum total cost if the firm want to produce something!
[Ans: $(x, y) = (0,0)$ is not considered because the firm want to produce something. The optimum total cost is 57,500 when the firm produces 75 of *X* and 125 of *Y*.]

13. Use **the Lagrange multiplier method** to determine the optimum values for each constrained optimisation problem and solve for the value of the corresponding Lagrange multiplier.

(a) $U = x^2 + 2x + 3z^2 - 6z + xz$ subject to $2x + 2z = 32$

[Ans: $(x, z) = (12,4), \lambda = 15$]

(b) $U = \frac{1}{2}a^2 + 4b^2 - 4a + 8c^2$ subject to $\frac{1}{2}a + 3b + c = 25$

[Ans: $(a, b, c) = (12,6,1), \lambda = 16$]

(c) $U = \ln(x + y)$ subject to $xy = 16$

[Ans: $(x, y) = (4,4), \lambda = \frac{1}{32}$]

14. Use **the Lagrange multiplier method** to determine the extreme values of the function

$$Z = x^2 + 2xy + w^2$$

subject to the two constraints

$$2x + y + 3w = 24 \text{ and } x + w = 8$$

[Ans: $(w, x, z) = (6,2,2); \lambda_1 = 4; \lambda_2 = 0$]

15. Maximise $f(x, y) = 2x + y - xy$ subject to $x + 2y \leq 10$. Find the maximum value of $f(x, y)$. Be sure to check that these optimum solutions satisfy all the complementary slackness conditions.

[Ans: Critical point is $(1,2)$ but $\left(\frac{7}{2}, \frac{13}{4}\right)$ is not because $\mu < 0$. $f(x, y)$ is maximised when $x = 1, y = 2$ and $f(1,2) = 2$]

16. The Cobb-Douglass production function of a new product is given by $N(x, y) = 2x^{0.5}y^{0.5}$ where *x* is the number of units of labour and *y* is the number of unit of capital required to produce *N*(*x*,*y*) units of the product. Each unit of labour costs 4 THB and each unit of capital costs 12 THB. If 72 THB has been budgeted for the production of this product ($4x + 12y = 72$). In addition, the number of units of labour is limited to 16 due to the limitation of space ($x \leq 16$). How should this amount be allocated between labour and capital in order to maximise production?
[Ans: *x* = 9 and *y* = 3]

17. Maximise $f(x, y) = xy + 2y$ subjected to $2x + y = 30$ and $y \leq 15$.

[Ans: $f(7.5,15) = 142.5$ is maximised., $f(6.5,17) = 144.5$ is not since $y > 15$]

18. Find the maximum value of the function

$$h(x, y) = 2x^2 - y^2$$

subject to the constraint that

$$x^2 + y^2 = 1$$

and the constraint that *x* and *y* are nonnegative. Be sure to check that these optimum solutions satisfy all the complementary slackness conditions.

Hint: We can use Extreme value theorem or Lagrange multiplier method to solve.
[Ans: $L = 2x^2 - y^2 - \lambda(x^2 + y^2 - 1) - \mu_1 x - \mu_2 y$, there are 9 equations to solve for 6 unknowns. The optimum solution is *x* = 1, *y* = 0 when $\mu_1 = 0$.]

19. Find critical point(s) of $f(x, y) = 6x + 4e^{-x} + e^y - 0.5e^{2y}$ subject to $x + y = 0$

(a) Use the substitution method

(b) Use the Lagrange multiplier method

[Ans: $(x, y) = (-\ln 3, \ln 3)$ and $(x, y) = (-\ln 2, \ln 2)$