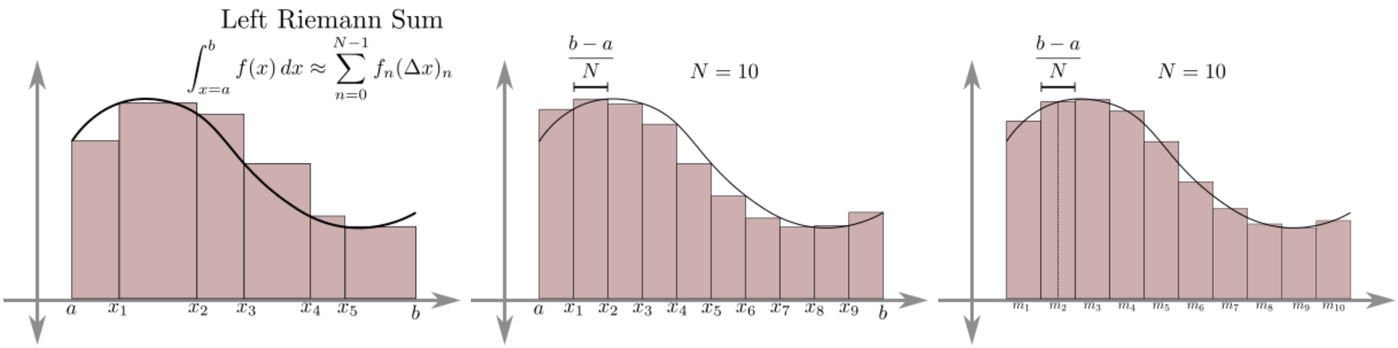
Definite Integral: Area Problem

1 The definite Integral: Introduction

Area problems can be used to define definite integral.
Area under graph: **find the area bounded by the x-axis and the graph of a continuous non-negative function $y = f(x)$ defined on an interval [a,b].**

1.1 Riemann sums

In the area problem, we consider the area bounded between the x-axis and the graph of a **continuous non-negative** function $y = f(x)$. Here, we consider **Riemann sums** of a function f that is defined on the closed interval $[a, b]$ (f can be discontinuous and it could be **negative**).



Definition 1.1 (Riemann sums)**.** Let f be a function defined on $[a, b]$. We can compute the Riemann sums by using the following 4 steps.

- Let n be the number of subintervals in $[a, b]$ and suppose that we have

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

  and the subinterval $[x_{k-1}, x_k]$ has the width given by $\triangle x_k = x_k - x_{k-1}, \qquad k = 1, \ldots, n.$

- Let $\|P\|$ denote the largest number of the n subinterval widths:

$$\|P\| = \max\{\triangle x_1, \triangle x_2, \ldots, \triangle x_n\}$$

  $\|P\|$ is called the norm of the partition P.

- Chose **sample points** $x_k^* \in [x_{k-1}, x_k]$, $k = 1, \ldots, n$.
- Form the **Riemann sum**:

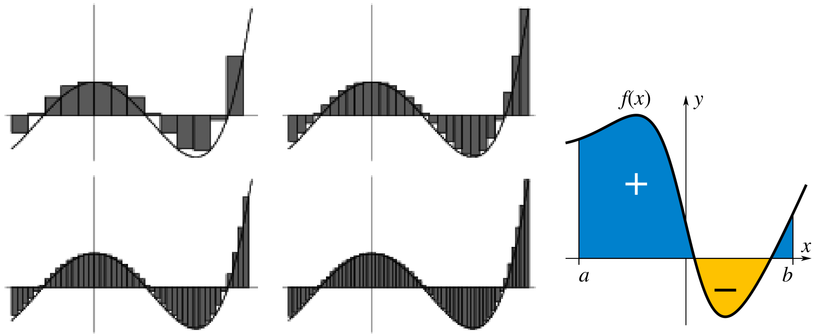
$$\sum_{k=1}^{n} f(x_k^*)\triangle x_k = f(x_1^*)\triangle x_1 + f(x_2^*)\triangle x_2 + \cdots + f(x_n^*)\triangle x_n.$$

Example 1.1. (Exercise) Compute the Riemann sum for $f(x) = x^2 + 1$ on $[-1, 2]$ with four subintervals determined by $x_0 = -1, \quad x_1 = -0.3, \quad x_2 = 0.5, \quad x_3 = 1, \quad x_4 = 2,$
with $x_1^* = -0.5, \quad x_2^* = 0, \quad x_3^* = 0.8, \quad x_4^* = 1.5,$ Find the norm of the partition.

[Ans: norm of the partition $\|P\| = \triangle x_4 = 1$]

1.2 Definite Integral: Definition

We can define definite integral by using the notion of Riemann sums with $\|P\|$ approaches 0.



Definition 1.2 (The Definite Integral)**.** Let f be a function defined on $[a, b]$. By using the notations in Definition 1.1, the **definite integral of f from a to b**, denoted by $\int_a^b f(x)dx$, is defined to be

$$\int_a^b f(x)\ dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\triangle x_k.$$

If the limit above exists, the function f is said to be **integrable** on $[a, b]$.
The number a is called **lower limits of integration** and
the number b is called **upper limits of integration**.
The function f is called the **integrand**.

In practice, we can compute the definite integral by using regular partition as follows.
Let x_k^* be the right endpoint. Then $x_k^* = a + k\triangle x_k$. Let $\triangle x_k = \triangle x = \frac{b-a}{n}$, for all $k = 1, \ldots, n$. Then

$$x_k^* = a + k\frac{b-a}{n} \qquad k = 1, 2, \ldots, n,$$

and as $\|p\| \to 0$, we have $n \to \infty$. So, the formula in Definition 1.2 is now written in term of n:

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}.$$

We now consider conditions that can guarantee the integrability of a function f.

Theorem 1.1 (Integrability)**.** Each of the followings is a sufficient condition for f to be integrable.

- If f is continuous on $[a, b]$, then $\int_a^b f(x)\ dx$ exists, i.e. f to be integrable on $[a, b]$.
- If (i) f is bounded on $[a, b]$: $\quad -B \le f(x) \le B, \quad \forall x \in [a, b]$ for some positive constant B, and (ii) f has a finite number of discontinuities on $[a, b]$, then $\int_a^b f(x)\ dx$ exists, i.e. f to be integrable on $[a, b]$.

The area bounded between the x-axis and the graph of a **continuous non-negative** function $f(x)$ can be written in term of the definite integrals as shown in the following theorem.

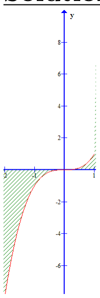
Theorem 1.2 (Area as a Definite Integral)**.** If f is continuous on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$, then the **area** A **under the graph** $y = f(x)$ on the interval $[a,b]$ can be given by

$$A = \int_a^b f(x)\ dx.$$

Example 1.2. Evaluate the following definite integral by using Definition 1.2 with regular partition:

$$\int_{-2}^{1} x^3 dx.$$

<u>Solution:</u> Set $a = -2$, $b = 1$, and $f(x) = x^3$.



$$\begin{aligned}
\int_{-2}^{1} x^3 dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left[f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[f\left(-2 + k\frac{1-(-2)}{n}\right)\frac{1-(-2)}{n}\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[f\left(-2 + \frac{3k}{n}\right)\frac{3}{n}\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(-2 + \frac{3k}{n}\right)^3\frac{3}{n}\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(-8 + 36\left(\frac{k}{n}\right) - 54\left(\frac{k}{n}\right)^2 + 27\left(\frac{k}{n}\right)^3\right)\frac{3}{n}\right] \\
&= \lim_{n\to\infty} \left[\left(-8\sum_{k=1}^{n} 1 + \frac{36}{n}\left(\sum_{k=1}^{n} k\right) - \frac{54}{n^2}\left(\sum_{k=1}^{n} k^2\right) + \frac{27}{n^3}\left(\sum_{k=1}^{n} k^3\right)\right)\frac{3}{n}\right] \\
&= \lim_{n\to\infty} \left[\left(-8n + \frac{36}{n}\left(\frac{n(n+1)}{2}\right) - \frac{54}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{27}{n^3}\left(\frac{n^2(n+1)^2}{4}\right)\right)\frac{3}{n}\right] \\
&= -\frac{15}{4}
\end{aligned}$$

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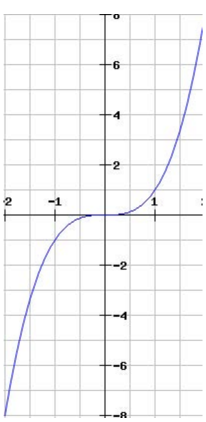
Example 1.3. Given that $\int_{-2}^{1} x^3 dx = -\frac{15}{4}$ and $\int_0^1 x^3 dx = \frac{1}{4}$, find the area between the graph $y = x^3$ and the x-axis on the interval $[-2, 0]$.



Theorem 1.3 (**Properties of Definite Integrals).** Let f and g be integrable functions. Let $a, b \in \mathbb{R}$ be constants.

1. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
   We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2. $\int_a^a f(x)\ dx = 0$
   If the upper and lower limits are the same then there is no work to do, the integral is zero.

3. $\int_a^b c\ f(x)\ dx = c\int_a^b\ f(x)\ dx$ where c is any number.
   So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4. $\int_a^b\ [f(x) \pm g(x)]\ dx = \int_a^b\ f(x)dx \pm \int_a^b\ g(x)]\ dx$
   We can break up definite integrals across a sum or difference.

5. $\int_a^b\ f(x)\ dx = \int_a^c\ f(x)\ dx + \int_c^b\ f(x)\ dx$ where c is any number.
   This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, [a,c] and [c,b]. Note however that c **does not need to be between** a **and** b.

6. $\int_a^b\ f(x)\ dx = \int_a^b\ f(t)\ dt$
   The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral wont affect the answer.

Example 1.4. Given that $\int_2^{-1}\ f(x)\ dx = 5$ and $\int_{-1}^2\ g(x)\ dx = -3$, determine the values of the following definite integrals.

1. $\int_{-1}^2\ 4f(x)\ dx$
2. $\int_{-1}^2\ 2f(x) + 3g(x)\ dx$
3. $\int_2^2\ f(x)^2 + \frac{g(x)}{|f(x)|+1}\ dx$

2 Fundamental Theorem of Calculus

Fundamental Theorem of Calculus provides a connection of the definite integral with the concept of anti-derivatives for continuous functions. It gives us a convenient way to evaluate the definite integral.

Theorem 2.1 (Fundamental Theorem of Calculus- Anti-derivative Form). Let f be a continuous function on $[a, b]$ and let F be any anti-derivative of f on $[a, b]$, i.e. $\frac{d}{dx}F(x) = f(x)$. Then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Example 2.1. Evaluate

$$\int_{-1}^{2} x^3\ dx.$$

Example 2.2. Evaluate

$$\int_{-1}^{1} x^4 + 3x + 1 - e^x\ dx.$$

Example 2.3. Evaluate

$$\int_0^{\pi} [\cos(x) - \sin(x)]\ dx.$$

Theorem 2.2 (**Fundamental Theorem of Calculus- Derivative Form).** Let f be a continuous function on $[a, b]$ and let the number $x \in [a, b]$. Then $g(x) = \int_a^x f(t)\ dt$ is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$, i.e.

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x).$$

Example 2.4. Use the Fundamental theorem of calculus to find

(a) $\frac{d}{dx}\int_{-1}^x e^t\ t^\pi\ dt.$

(b) $\frac{d}{dx}\int_{-3}^x \frac{1}{\sqrt[3]{t^4+2}}\ dt.$

Example 2.5. Find $\frac{d}{dx}\int_x^0 \frac{\cos(t)}{t^2+1}\ dt.$

Example 2.6. Find $\frac{d}{dx}\int_\pi^{x^3} \cos(t)\ dt.$

Let $u = x^3$.

$$\frac{d}{dx}\int_\pi^{x^3} \cos(t)\ dt \quad = \quad \frac{d}{du}\left(\int_\pi^u \cos(t)\ dt\right)\frac{du}{dx}$$

Example 2.7. Find $\frac{d}{dx}\int_{e^x}^{x^2} t^2\sin(t)\ dt.$

Integrate a piecewise continuous functions

A function f is called **piecewise continuous** on $[a,b]$ if there are a finite number of points c_k, $k = 0,1,2,\ldots,n$ with

$$a = c_0 < c_1 < \cdots < c_{n-1} < c_n = b,$$

such that f is continuous on each interval (c_{k-1}, c_k), $k = 1,\ldots,n$. Then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \cdots + \int_{c_{n-1}}^{b} f(x)\,dx.$$

Example 2.8. Let $f(x) = \begin{cases} x-1, & x \in (-\infty, 0) \\ x^2, & x \in [0,2) \\ 3x+1, & x \in [2,\infty). \end{cases}$ Evaluate $\int_{-1}^{3} f(x)\,dx$.

Example 2.9. Evaluate $\int_{-1}^{3} |x-1|\,dx$.

3 Substitution Rule for Definite Integrals

Recall the technique u-**substitution** for indefinite integral. We can use a similar approach to evaluate definite integral as follows.

Theorem 3.1 (Substitution in a Definite Integral). Let $u = g(x)$ be a function that has a continuous derivative on $[a, b]$ and let f be a function that is continuous on the range of g. If $F'(u) = f(u)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx \quad = \quad \int_{g(a)}^{g(b)} f(u)\, du \quad = \quad F(d) - F(c) \qquad \text{where } c = g(a), \quad d = g(b).$$

Example 3.1. Evaluate $\int_1^3 x\sqrt[3]{x^2 - 1}\, dx$.

Example 3.2. Evaluate $\int_3^5 \frac{4t}{2-8t^2}\ dt$.

Let f be a function defined on an interval I. f is called an **even function** if

$$f(-x) = f(x), \forall x \in I,$$

and f is called an **odd function** if

$$f(-x) = -f(x), \forall x \in I.$$

Theorem 3.2 (**Even Function Rule**). Let f be an even function and f is integrable on $[-a, a]$ where a is a constant number, then

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x).$$

Theorem 3.3 (**Odd Function Rule**). Let f be an odd function and f is integrable on $[-a, a]$ where a is a constant number, then

$$\int_{-a}^{a} f(x)\ dx = 0.$$

Example 3.3. Evaluate $\int_{-\pi}^{\pi} (x^5 + \sin(x))^3\ dx$.

4 Exercise

1. Evaluate the given integrals.

   (a) $\int \frac{(x-3)^2}{\sqrt{x}}\ dx$

   (b) $\int \left[e^x + \frac{1}{x} - \sin(x) + 3\sinh(x) + \pi^2 + \csc^2(x) - \sec(x)\tan(x)\right]\ dx$

   (c) $\int \sqrt{2+3x}\ dx$

   (d) $\int \frac{1-x}{\sqrt{3-x^2}}\ dx$

   (e) $\int \left[\sin^3(x)\cos(x) + \frac{\sec^2(x)}{\sqrt{\tan(x)}}\right]\ dx$

   (f) $\int_0^{\frac{\pi}{2}} \cos(x) + \cos^5(x)\sin(x)\ dx$

   (g) $\int_{-1}^{1} \left(\frac{x^5+x}{(x^4+4x^2+4)^3} + |x|\right)\ dx$

   (h) $\int_{\frac{1}{2}}^{\frac{e}{2}} \frac{[\ln(2x)]^5}{x}\ dx$

2. Determine the function $f(x)$ such that $f''(x) = (1+x)^4$, $f'(0) = 0$ and $f(0) = 0$.

3. Use the Fundamental Theorem of Calculus (derivative form)to find the indicated derivative.

   (a) $\frac{d}{dx}\int_2^x \left[t^2e^t + \ln(|\sin(t)|)\right]\ dt$

   (b) $\frac{d}{dx}\int_{\ln(x)}^{\sin(x)} \frac{1}{1+t^5}\ dt$