CHAPTER 7

22 oct.

Derivatives of More-Than-One Independent Variable Function

26 oct.

Topics:

- First-order partial derivatives
- Second-order partial derivatives
- Differential
- Total differential
- Total derivatives
- Implicit function and its derivative
- Examples in economics
    - Partial market equilibrium
    - Multipliers in macroeconomic models
    - Utility function
    - Production function
    - Etc.

In comparative-static analysis, we are likely to encounter the situation in which several parameters/independent variables appear in a model. Many functions in economics, such as Production function, Cost function, Profit function, Utility function, Demand function, deal with many independent variables. Therefore, we must learn how to find the derivative of a function of more than one variable (Multivariable calculus).

"Transition of idea"

Ch 5, 6	Ch 7 Multivariable calculus
$y = f(x)$	$y = f(x_1, x_2, \ldots, x_n)$
$\frac{dy}{dx}$ "derivatives"	partial derivatives $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n}$
	total differentials dy
	total derivatives $\frac{dy}{dx_1}, \frac{dy}{dx_2}, \ldots, \frac{dy}{dx_n}$
$\pi(Q)$, $Q(L)$, $U(x)$	$\pi(Q_1, Q_2)$, $Q(K, L)$, $U(x_1, x_2)$
$\max_{Q} \pi(Q)$	$\max_{Q_1, Q_2} \pi(Q_1, Q_2)$
FOC: $\pi'(Q) = 0$	FOC: Jacobian matrix
SOC: $\pi''(Q) < 0$	SOC: Hessian matrix

Examples of multivariable function in economics

The constant elasticity demand function

$$q_1 = f(p_1, p_2, y) = kp_1^{\alpha}p_2^{\beta}y^{\gamma}$$

E_{p_1}, E_{p_2}, E_y will be constant, equal to α, β, γ, respectively

The firm's production function

Linear:

$$q = a_1x_1 + a_2x_2$$

Cobb-Douglas:

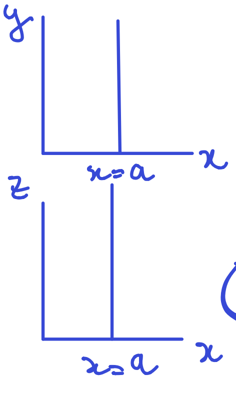
$$q = kx_1^{\alpha}x_2^{\beta}$$

Leontief:

$$q = min\left\{\frac{x_1}{c_1}, \frac{x_2}{c_2}\right\}$$

Constant elasticity of substitution:

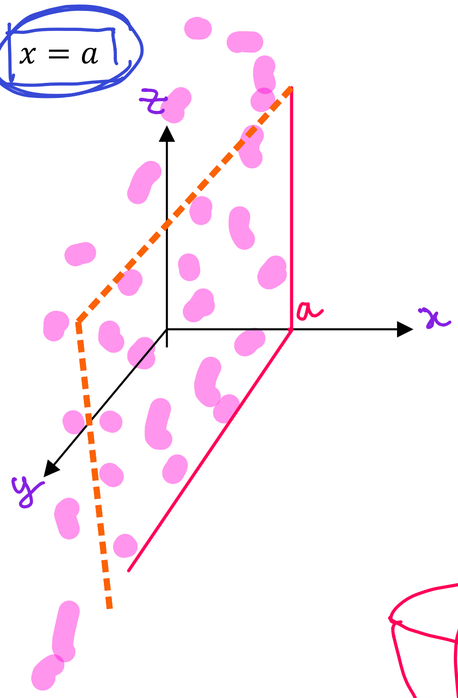
$$q = C\left[\pi x_1^{\frac{\sigma-1}{\sigma}} + (1-\pi)x_2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

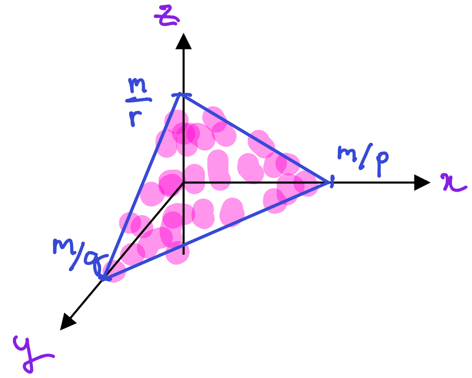


Geometric Representations of Functions of ***Several*** *Variables*

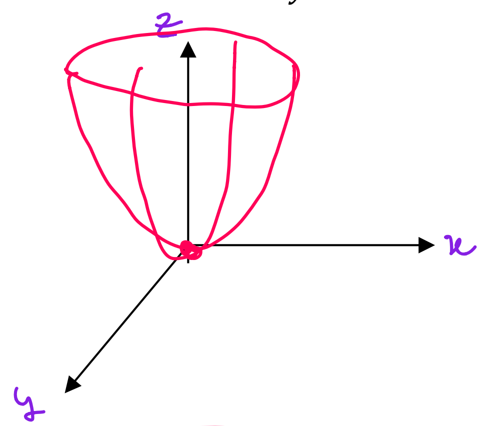
$x = a$

$px + qy + rz = m$



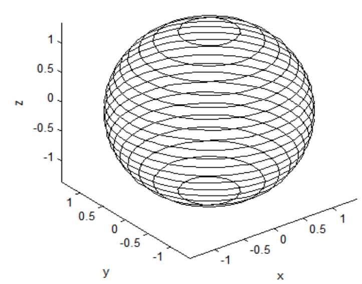


$z = x^2 + y^2$

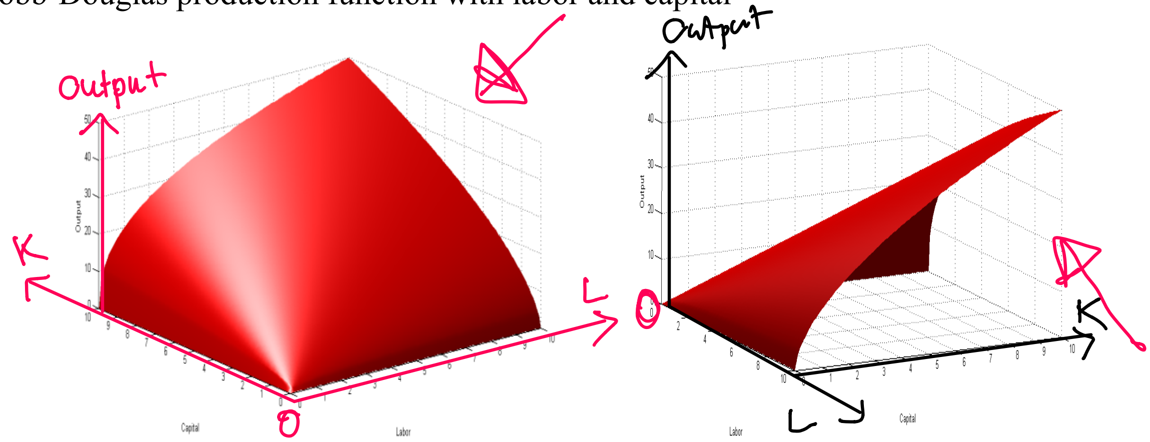


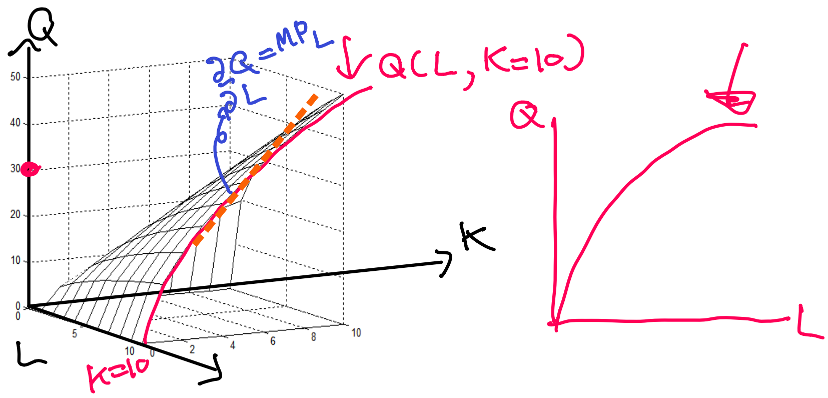
$x^2 + y^2 + z^2 = 2$

x^2+y^2+z^2 = 2



Cobb-Douglas production function with labor and capital

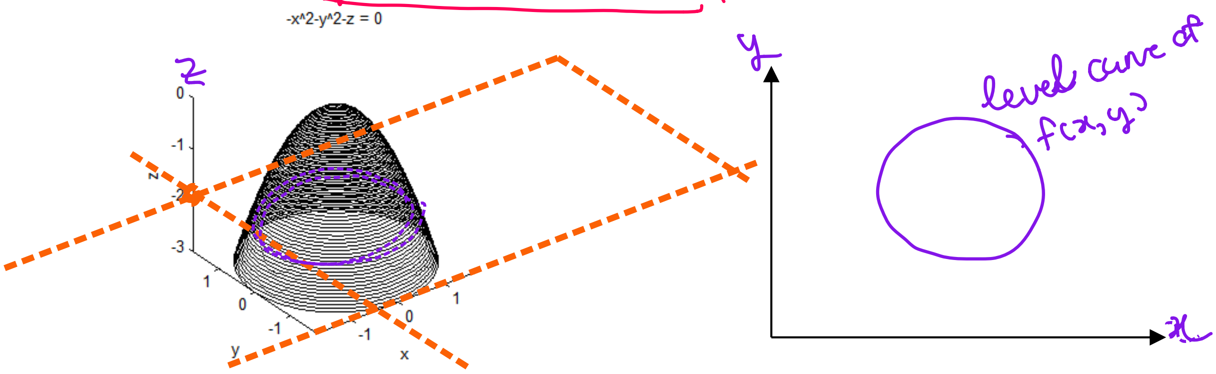




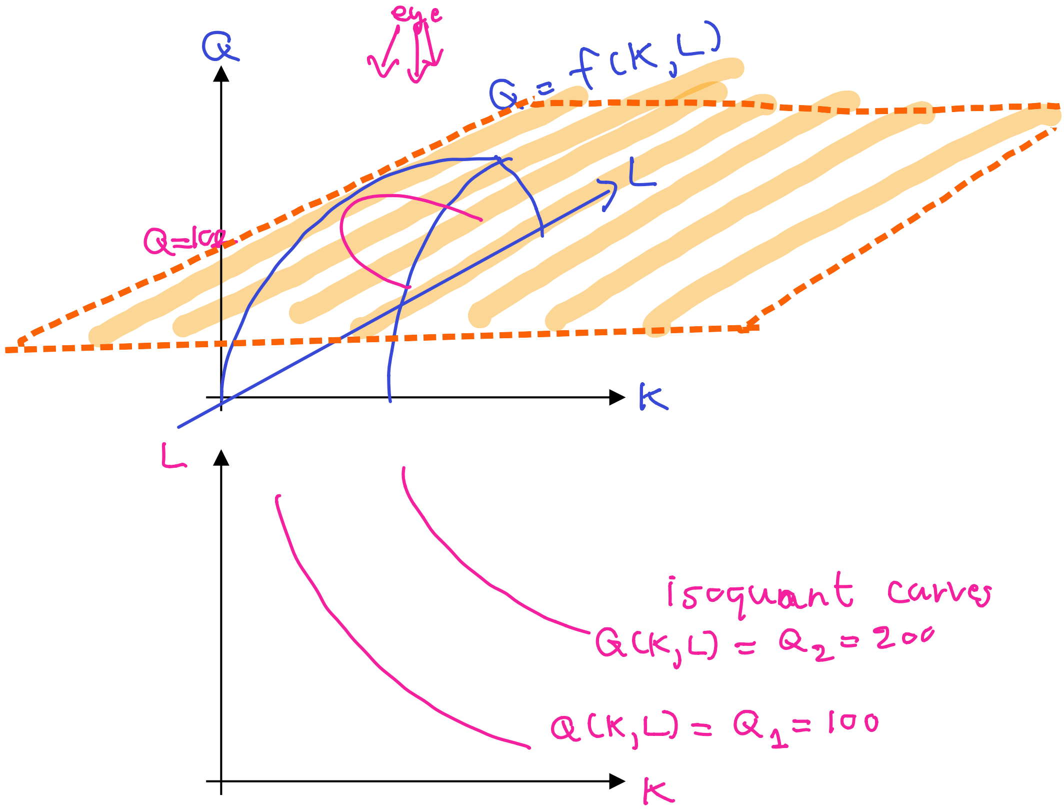
Level Curves for z = f(x, y)

From above three-dimensional graphs, we can find the relationship between x and y at each level of function f. That is, we can find x and y such that $f(x,y) = c$. Looking at each level of f and plot the relevant x and y, then we will get “the level curve” at that level of f.

Draw the level curve of $z = f(x,y) = -x^2 - y^2$



A level curve of production function is an isoquant curve.



Partial Derivative

Let

$$y = f(x_1, x_2, \dots, x_n),$$

where $x_1, \dots, x_n$ are independent variable and all independent of one another, so that each can vary by itself without affecting the others. If the variable x_1 changes by Δx_1, while $x_2, \dots, x_n$ all remain fixed, there will be a corresponding change in y, Δy. The difference quotient is:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1}$$

The partial derivative of y with respect to x_1 is:

$$\lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1} \equiv \frac{\partial y}{\partial x_1} \equiv f_1$$

The key 🔑: For n independent variables,

we MUST hold (n-1) independent variable constant, while allowing for one variable to vary.

$y = f(x_1, x_2)$

Find $\frac{\partial y}{\partial x_1}$ by assuming x_2 is a constant

$\frac{\partial y}{\partial x_2}$ x_1

Example: $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, what are $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}$?

$\frac{\partial y}{\partial x_1} = 6x_1 + x_2$

$\frac{\partial y}{\partial x_2} = x_1 + 8x_2$

→ Example: $y = f(u, v) = (u + 4)(3u + 2v)$, what are $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$?

→ Example: $y = f(u, v) = (3u - 2v)/(u^2 + 3v)$, what are $\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$?

H.W.

- *Compute all the partial derivatives of the following function*
  a) $4x^2y - 3xy^3 + 6x$
  b) xy
  c) xy^2
  d) e^{2x+3y}
  e) $\frac{x+y}{x-y}$
  f) $3x^2y - 7x\sqrt{y}$

- *Compute the partial derivative of the Cobb-Douglas function*

  $q = k_1 x_1^{a_1} x_2^{a_2}$ *and of the Constant Elasticity of Substitution (CES)*

  production function $q = C\left[\pi x_1^{\frac{\sigma-1}{\sigma}} + (1-\pi)x_2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$

Economic Interpretation

Production function $F(K,L)$

$\frac{\partial F(K,L)}{\partial K}$ is Marginal product of capital, MP_K, given labor as a constant

$\frac{\partial F(K,L)}{\partial L}$ is Marginal product of labor, MP_L, " capital

Example: Find MP_K and MP_L of Cobb-Douglas production function $Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$, at the current level of factors at $L = 625$, $K = 10{,}000$

$MP_L = \frac{\partial Q}{\partial L} = 4K^{3/4} \cdot \frac{1}{4} L^{\frac{1}{4}-\frac{4}{4}} = K^{3/4} L^{-3/4} = \left(\frac{K}{L}\right)^{3/4}$ (K: just like a constant)

$MP_K = \frac{\partial Q}{\partial K} = 4L^{1/4}\, \frac{3}{4} K^{\frac{3}{4}-\frac{4}{4}} = 3K^{-1/4} L^{1/4}$

$MP_L \big|_{L=625, K=10{,}000} = (10{,}000)^{3/4} (625)^{-3/4} = 16^{3/4} = 8$

$MP_K \big|_{L=625, K=10{,}000} = 3(10{,}000)^{-1/4} (625)^{1/4} = \frac{3}{2}$

Example: Find MU_{x_1}, MU_{x_2} of the utility function $U(x_1, x_2) = x_1 log x_2$

$MU_{x_1} = \frac{\partial U}{\partial x_1} = \frac{\partial}{\partial x_1}\left[(\log x_2) x_1\right] = \log x_2$ (constant)

$MU_{x_2} = \frac{\partial U}{\partial x_2} = \frac{x_1}{x_2 \ln 10}$

note that: $y = \log_a x \rightarrow \frac{dy}{dx} = \frac{1}{x \ln a}$

Higher-Order Partial Derivative

Since $\frac{\partial f}{\partial x_1}$ is a function of $x_1, \ldots, x_n$, we can also find partial derivative of $\frac{\partial f}{\partial x_1}$

The second order partial derivative of f is:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

young's theorem: $= \frac{\partial^2 f}{\partial x_i \partial x_j}$

$$\frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2}$$

$\frac{\partial^2 f}{\partial x_j \partial x_i}, i \neq j$, is called cross partial derivatives or mixed partial derivatives.

Second-order Derivative and Hessians

Example: Find all second derivatives of $Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$

1st partial derivative: (Jacobian Matrix)

$Q_K = \frac{\partial Q}{\partial K} = 3K^{-1/4}L^{1/4}$ (How Q changes when K changes, holding L constant)

$Q_L = \frac{\partial Q}{\partial L} = K^{3/4}L^{-3/4}$

2nd order partial derivative:

$Q_{KK} = \frac{\partial Q_K}{\partial K} = \frac{\partial^2 Q}{\partial K^2} = -\frac{3}{4}K^{-5/4}L^{1/4}$

How MP_K changes, when K changes, holding L constant

$Q_{KL} = \frac{\partial Q_L}{\partial K} = \frac{\partial^2 Q}{\partial K \partial L} = \frac{3}{4}K^{-1/4}L^{-3/4}$

$Q_{LK} = \frac{\partial Q_K}{\partial L} = \frac{\partial^2 Q}{\partial L \partial K} = \frac{3}{4}K^{-1/4}L^{-3/4}$

How MP_K changes, when L changes, holding K constant

$Q_{LL} = \frac{\partial Q_L}{\partial L} = \frac{\partial^2 Q}{\partial L^2} = -\frac{3}{4}K^{3/4}L^{-5/4}$

→ 2x2 matrix, Hessian matrix

note: $Q_{LK} \overset{Young}{=} Q_{KL}$

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Function with n independent variables will n^2 second order partial derivatives, from which we can write out the $n \times n$ matrix. Row i, Column j corresponds to $\frac{\partial^2 f}{\partial x_i \partial x_j}$. This matrix is called Hessian Matrix. The Hessian matrix is a symmetric matrix (Young's theorem).

$$H = D^2 f_x = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

H.W.

1) Let $y = f(x_1, x_2) = x_1 e^{x_1 + x_2^2}$ find f_1 , f_2, and Hessian Matrix

2) Let $Q = f(K, L, T) = AK^{\alpha} L^{\beta} T^{\gamma}$, find marginal products, and Hessian matrix

The Total Differential of A Function of Several Variables

Derivatives vs. Differentials

The symbol $\frac{dy}{dx}$ for the derivative of the function $y = f(x)$ has been regarded as a single entity.

We shall now reinterpret $\frac{dy}{dx}$ as a ratio of two quantities, dy and dx, the differentials of y and x, respectively.

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)dx$$

$f'(x)$ is a “converter” translating a given independent change dx into a counterpart change in dependent change dy.

The process of finding dy from a given function $y = f(x)$ is called differentiation.

The process of finding derivative $\frac{dy}{dx}$ from a given function $y = f(x)$ is called differentiation with respect to x.

H.W. Given that $y = 3x^2 + 7x - 5$, *find* dy

Total differentials

The concept of differential can easily be extended to a function of two or more independent variable. Consider a saving function:

$$S = S(Y, i)$$

S is Saving, Y is national Income, i is Interest rate.

The partial derivative of S with respect to Y: ……………………………………………

For any change in Y, dy, the resulting change in S can be approximated by the quantity: …………………………………………

The partial derivative of S with respect to i: ……………………………………………

For any change in i, di, the resulting change in S can be approximated by the quantity: …………………………………………

The total change in S is then approximated by **the total differential of saving function**:

$ds =$ ……………………………………………………………………………………

The partial differential of the saving function:

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Rule of Differentials

For a constant k, and function $U(X_1, X_2)$ and $V(X_1, X_2)$

Rule 1: $dk = 0$
Rule 2: $d(cU^n) = cnU^{n-1}dU$
Rule 3 : $d(U \pm V) = dU \pm dV$
Rule 4 : $d(UV) = UdV + VdU$
Rule 5: $d\left(\frac{U}{V}\right) = \frac{VdU - UdV}{V^2}$

H·W· Find dy *for*

1) $y = 5x_1^2 + 3x_2$

2) $y = 3x_1^2 + x_1x_2^2$

3) $y = \frac{x_1 + x_2}{2x_1^2}$

4) $y = 3x_1(2x_2 - 1)(x_3 + 5)$

5) $y = -5 + 30x_1 - 3x_1^2 + 25x_2 - 5x_2^2 + x_1x_2$, *given that* $x_1 = 5,\ x_2 = 2, dx_1 = 0.02,\ dx_2 = 0$

Total Derivatives

Question: Consider equilibrium consumption function which is a function of equilibrium national income and exogenous tax, $C(Y^*, T_0)$. What is the rate of change of the function $C(Y^*, T_0)$ with respect to T_0, when Y^* and T_0 are related?

To answer this question, we need to learn about total derivative.

Unlike a partial derivative, a total derivative does not require the argument Y^* to remain constant as T_0 varies. A total derivative allows for the postulated relationship between the two arguments.

Finding the Total Derivative

Consider $y = f(x, w)$, where $x = g(w)$

Question: What is the total derivative of y with respect to w? How to do the total differentiation of y with respect to w?

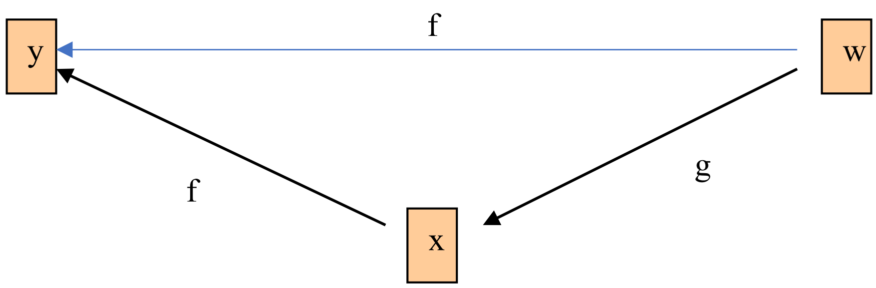
Note that: the two functions f and g can be combined into a composite function $\mathrm{y} = \mathrm{f}(\mathrm{g}(\mathrm{w}), \mathrm{w})$

The three variables $\mathrm{y}, \mathrm{x}, \mathrm{w}$ are related to one another. In the following *channel map, w,* the ultimate source of change, can affect y through two separate channels:

(1) Directly, via the function f

(2) Indirectly, via the function g, then f

“Channel Map”



The direct effect can be represented by the partial derivative

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The indirect effect can be represented by a product of two derivatives:

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, by the chain rule for a composite function

Adding up the two effects give us the total derivative of y with respect to w:

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Alternatively, we can also find the total derivative by:

(a.) Total differentiating the function $\boldsymbol{y = f(x, w)}$

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(b.) Dividing through the total differential by dw

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Example 1: Find Total derivative $\frac{dy}{dw}$, when
$y = f(x, w) = 3x - w^2, x = g(w) = 2w^2 + w + 4$

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Example 2: $U = u(c, s), and\ s = g(c)$, c is quantity of coffee, s is quantity of sugar, which depends on quantity of coffee consumed. How much will total utility change as quantity of coffee consumed changes?

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Three channels:

$y = f(x_1, x_2, w),\ x_1 = g(w), x_2 = h(w)$ What is $\frac{dy}{dw}$?

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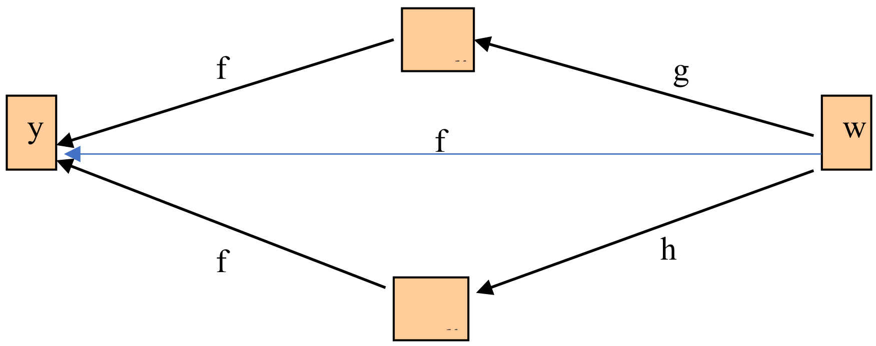

Example 3: $Q = Q(K, L, t),\ K = K(t),\ L = L(t)$

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Example 4: $Q^D = g(P, I)$, $I = f(P)$, Q^D *is quantity demanded,* I *is income, and* P *is price.*

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H.W. Find total derivative of

(1.) $z = f(x, y, t), x = a + bt, y = c + dt$

(2.) $z = f(x, y) = 2x + xy - y^2, x = g(y) = 3y^2$

Chain rule:

Example 5: Let $y = \ln(x_1 + x_2)$, $x_1 = t$, *and* $x_2 = t^2$. *Find* $\frac{dy}{dt}$ *by direct substitution, draw channel diagram, and also find* $\frac{dy}{dt}$ *by chain rule.*

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H.W. Find $\frac{dz}{dt}$ *at* $t = 0$, $z = \frac{5t^2+3xy}{2w^2\,y}$, $x = t^2 + 1$, $y = \sqrt{t^2 + 1}$, $w = e^t + 1$

Implicit Functions

A function given in the form of $y = f(x)$ is called an explicit function, because the variable y is explicitly expressed as a function of x. For example, $y = f(x) = 3x^4$.

Consider a given $F(y, x) = 0$. For $F(y, x) = 0$, the left hand side is a function of the two variables y and x.

For example, $y^3 - x^2y - \frac{1}{y} + 5xy = 0$.

$F(y, x) = 0$ can imply the function $y = f(x)$, in which case the function f is called an implicit function.

Derivatives of Implicit function

If the equation $F(y, x) = 0$ can be solved for y, we can write out the function $y = f(x)$ and find its derivatives by the methods learned before.

But if the equation $F(y, x) = 0$ cannot be solved for y, the derivatives of implicit function can be found by applying the concept of total differentiation.

$$F(y, x) = 0$$
$$dF(y, x) = d0$$
$$dF(y, x) = 0$$
$$F_y dy + F_x dx = 0$$

Thus, the implicit-function rule:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

For $F(y, x_1, \dots, x_m) = 0$, the partial derivative f_i of the implicit function $y = f(x_1, \dots, x_m)$ is:

$$f_i \equiv \frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y}$$

Example: Consider Utility function $U = U(x_1, x_2)$, find the marginal rate of substitution. And, if $U = U(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, Is the indifferent curve convex?

Homework: Find $\frac{dy}{dx}$ for when:

1.) $y^3 - x^2 y - \frac{1}{y} + 5xy = 0$

2.) $x^3 + 2x^2 y - 3xy^2 - y^3 = 0$

3.) $x^2 y^2 + e^x - e^y = 0$

4.) $3\ln(x + y) = y^2 - 4x^2$

Applications in Economics

Utility function

Consider Utility function $U = U(x_1, x_2)$. Find marginal utility of each good. What are the conditions for law of diminishing marginal utility to hold?

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Let $U(x_1, x_2) = (x_1 + 2)^2(x_2 + 3)^2$. Compare marginal utility of good 1 and good 2 when $x_1 = x_2 = 3$ units

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Production function

Consider $Q = f(K, L) = AK^{\alpha}L^{\beta}, 0 < \alpha, \beta < 1$

Find marginal product of each factor of production. What are the conditions for law of diminishing marginal product to hold?

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Consider $Q = f(K, L)$. Whether a production is constant, increasing and decreasing return to scale can be considered as follows.

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Elasticity

Elasticity of output with respect to factor of production

$Q = f(K, L) = AK^{\alpha}L^{\beta}, 0 < \alpha, \beta < 1$

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Elasticity of output with respect to factor of production

Consider $Q_A^D = f(P_A, P_B, y) = 100 - 10P_A + 15P_B + 0.3y$, Find own price/cross price/income elasticity of quantity demanded. Let $P_A = 8$, $P_B = 5$, $I = 500$, $Q_A = 60$

Own price elasticity:

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Cross price elasticity:

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Income elasticity:

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HOMEWORK

1.) Market equilibrium

$$Q_d = Q_s$$

$$Q_d = a - bP \quad (a, b > 0)$$

$$Q_s = -c + dP \quad (c, d > 0)$$

$$P^* = \frac{a+c}{b+d} \quad , \quad Q^* = \frac{ad-bc}{b+d}$$

What are $\frac{\partial P^*}{\partial a}, \frac{\partial Q^*}{\partial a}, \frac{\partial P^*}{\partial c}, \frac{\partial Q^*}{\partial c}, \frac{\partial P^*}{\partial b}, \frac{\partial Q^*}{\partial b}, \frac{\partial P^*}{\partial d}, \frac{\partial Q^*}{\partial d}$?

2.) **Multipliers in Keynesian crossing model**

$$Y = C + I + G + X - M$$

$$Y = C_0 + C_1\left(Y - T\right) + I_0 + iY + G_0 + X_0 + \gamma_0 Y^* - M_0 - \lambda_0 Y$$

$$Y = C_0 + C_1\left(Y - t_0 - t_1 Y\right) + I_0 + iY + G_0 + X_0 + \gamma_0 Y^* - M_0 - \lambda_0 Y$$

$$Y_E = \frac{C_0 - t_0 C_1 + I_0 + G_0 + X_0 + \gamma_0 Y^* - M_0}{1 - C_1\left(1 - t_1\right) - i + \lambda_0}$$

What are $\frac{\partial Y_E}{\partial Y^*}, \frac{\partial Y_E}{\partial I_0}, \frac{\partial Y_E}{\partial C_0}, \frac{\partial Y_E}{\partial C_1}$?