



MULTIPLE INTEGRALS

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The idea is to extend the idea of a definite integral to double and triple integrals of functions of two or three variables. These ideas are then used to compute volumes.

MULTIPLE INTEGRALS

Double Integrals over Rectangles

In this section, we will learn about:
Double integrals and using them
to find volumes and average values.

DOUBLE INTEGRALS OVER RECTANGLES

Just as our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid.

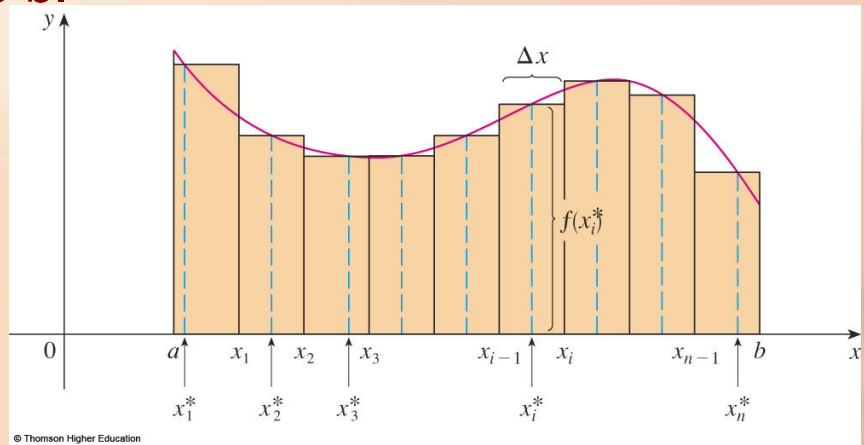
In the process, we arrive at the definition of a double integral.

DEFINITE INTEGRAL—REVIEW MA216

First, let's recall the basic facts concerning definite integrals of functions of a single variable.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Then, $\int_a^b f(x) dx$ represents the area under the curve $y = f(x)$ from a to b .



Partial Antidifferentiation

We can antidifferentiate a function of two or more variables with respect to one of the variables by treating all the other variables as though they are constants.

$\int f(x, y) dx \Rightarrow$ antidifferentiate $f(x, y)$ w.r.t. x ,
holding y constant.

$\int f(x, y) dy \Rightarrow$ antidifferentiate $f(x, y)$ w.r.t. y ,
holding x constant.

Example: Evaluate:

$$(A) \int (6xy^2 + 3x^2)dy \quad (B) \int (6xy^2 + 3x^2)dx$$

$$(A) 2xy^3 + 3x^2y + C(x) \quad (B) 3x^2y^2 + x^3 + E(y)$$

Example: Evaluate:

$$(A) \int (4xy + 12x^2y^2)dy \quad (B) \int (4xy + 12x^2y^2)dx$$

$$(A) 2xy^2 + 3x^2y^4 + C(x) \quad (B) 2x^2y + 4x^3y^3 + E(x)$$

Example: Evaluate:

$$(A) \int_0^2 (6xy^2 + 3x^2)dy \quad (B) \int_0^1 (6xy^2 + 3x^2)dx$$

$$(A) 16x + 6x^2 \quad (B) 3y^2 + 1$$

Example: Evaluate:

$$(A) \int_0^1 (4xy + 12x^2y^2)dy \quad (B) \int_0^3 (4xy + 12x^2y^2)dx$$

$$(A) 2x + 3x^2 \quad (B) 18y + 108y^3$$

Example:

Evaluate the iterated integrals.

a. $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

b. $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

ITERATED INTEGRALS

Example 1 a

Regarding x as a constant,
we obtain:

$$\begin{aligned} \int_1^2 x^2 y \, dy &= \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} \\ &= x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) \\ &= \frac{3}{2} x^2 \end{aligned}$$

ITERATED INTEGRALS**Example 1 a**

Thus, the function A in the preceding discussion is given by

$$A(x) = \frac{3}{2} x^2$$

in this example.

ITERATED INTEGRALS**Example 1 a**

We now integrate this function of x from 0 to 3:

$$\begin{aligned} \int_0^3 \int_1^2 x^2 y \, dy \, dx &= \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx \\ &= \int_0^3 \frac{3}{2} x^2 \, dx = \left. \frac{x^3}{2} \right]_0^3 \\ &= \frac{27}{2} \end{aligned}$$

ITERATED INTEGRALS**Example 1 b**

Here, we first integrate with respect to x :

$$\begin{aligned}\int_1^2 \int_0^3 x^2 y \, dx \, dy &= \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy \\ &= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\ &= \int_1^2 9y \, dy = 9 \left[\frac{y^2}{2} \right]_1^2 = \frac{27}{2}\end{aligned}$$

Example: Evaluate:

$$(A) \int_0^1 \left[\int_0^2 (6xy^2 + 3x^2) dy \right] dx \quad (B) \int_0^2 \left[\int_0^1 (6xy^2 + 3x^2) dx \right] dy$$

Example: Evaluate:

(A) $\int_0^3 \left[\int_0^1 (4xy + 12x^2y^3) dy \right] dx$

(B) $\int_0^1 \left[\int_0^3 (4xy + 12x^2y^3) dx \right] dy$

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Definition of the Double Integral (Volume)

In a similar manner, we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d]$$

$$= \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$.

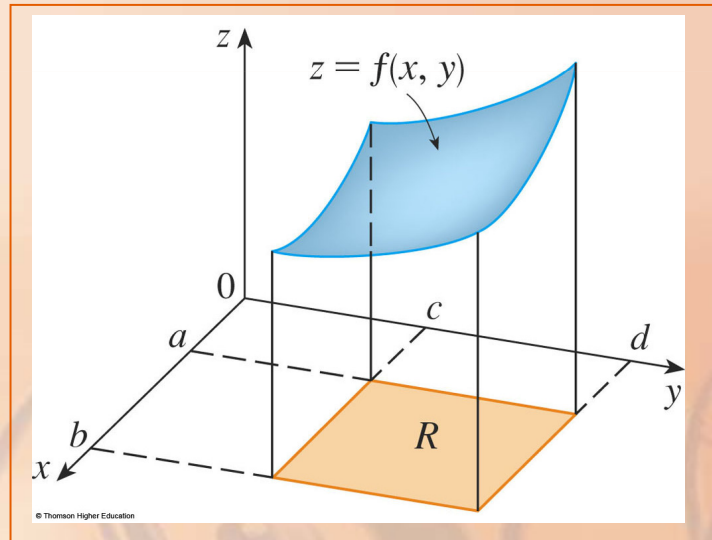
- The graph of f is a surface with equation $z = f(x, y)$.

VOLUMES

Let S be the solid that lies above R and under the graph of f , that is,

$$S = \{(x, y, z) \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$

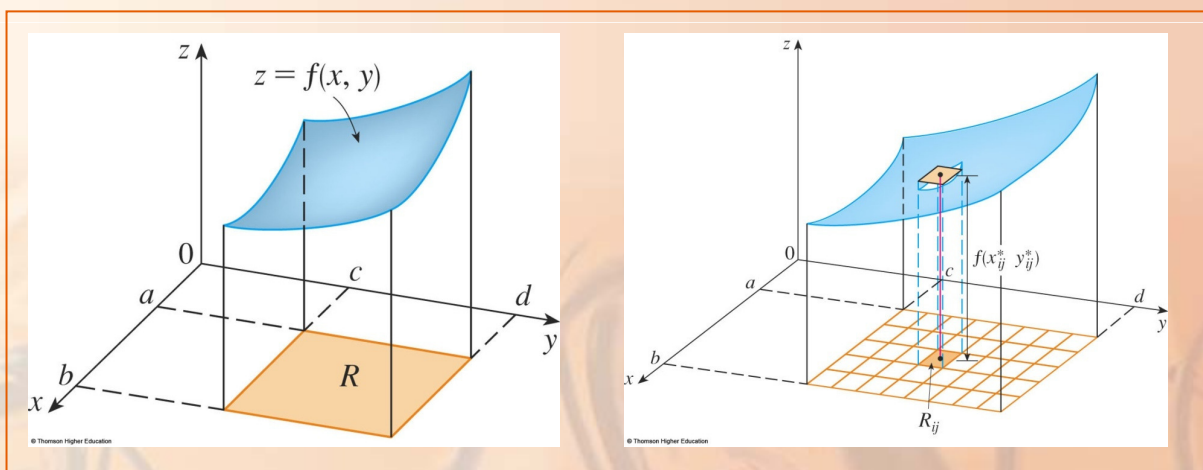
Our goal is to find the volume of S .



VOLUMES

The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*) \Delta A$$

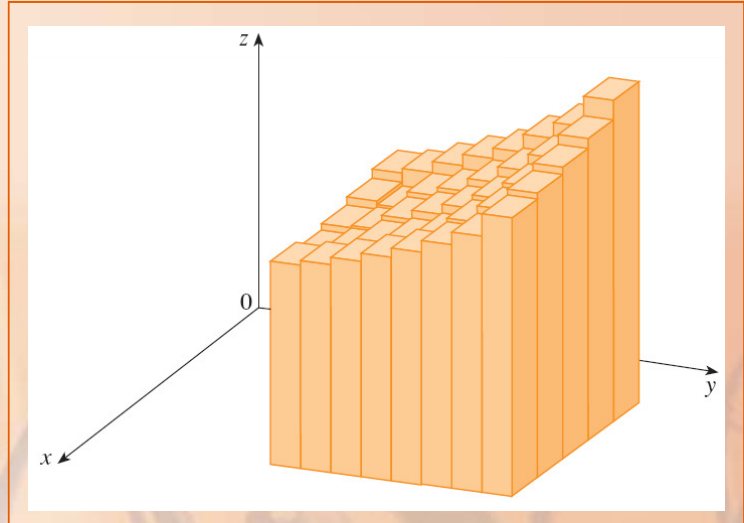


VOLUMES

Equation 3

Thus, we get an approximation to the total volume of S :

$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$



VOLUMES

Equation 4

Our intuition tells us that the approximation given in Equation 3 becomes better as m and n become larger.

So, we would expect that:

$$V = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

DOUBLE INTEGRAL

Definition 5

The double integral of f over the rectangle R is:

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

if this limit exists.

INTEGRABLE FUNCTION

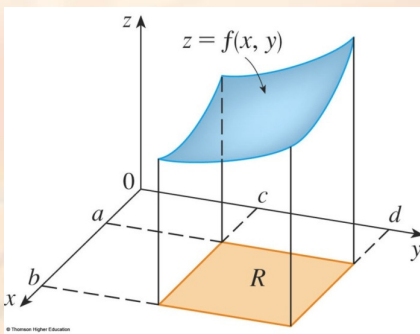
A function f is called integrable if the limit in Definition 5 exists.

- It is shown in courses on advanced calculus that all continuous functions are integrable.
- In fact, the double integral of f exists provided that f is “not too discontinuous.”

DOUBLE INTEGRAL

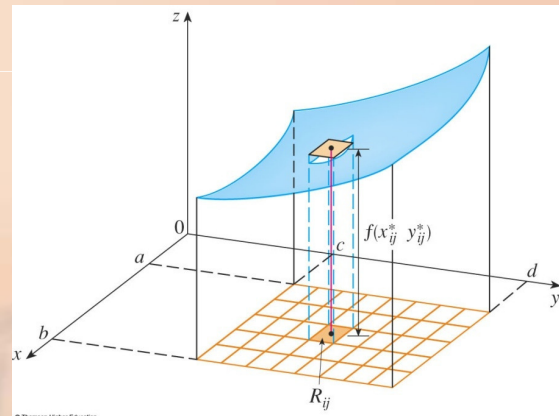
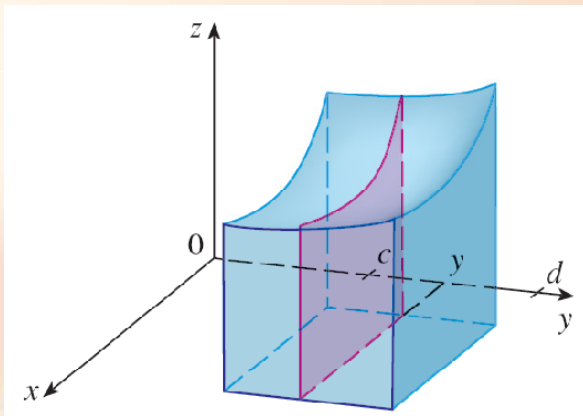
If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is:

$$\begin{aligned}\iint_R f(x, y) dA &= \int_a^b \left[\int_c^d f(x, y) dy \right] dx \\ &= \int_c^d \left[\int_a^b f(x, y) dx \right] dy\end{aligned}$$



DOUBLE INTEGRAL

$$\iint_R f(x, y) dA = \int_a^b \left[\int_c^d f(x, y) dy \right] dx = \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$



Example:

Evaluate the double integral

$$\iint_R (x - 3y^2) dA$$

where

$$R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$$

Example:

$$\iint_R (x - 3y^2) dA = \int_0^2 \int_1^2 (x - 3y^2) dy dx$$

$$= \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7) dx = \left[\frac{x^2}{2} - 7x \right]_0^2$$

$$= -12$$

Example:

This time, we first integrate with respect to x :

$$\begin{aligned}\iint_R (x - 3y^2) dA &= \int_1^2 \int_0^2 (x - 3y^2) dx dy \\ &= \int_1^2 \left[\frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} dy \\ &= \int_1^2 (2 - 6y^2) dy = 2y - 2y^3 \Big|_1^2 \\ &= -12\end{aligned}$$

Example:

Notice the negative answer in this example.

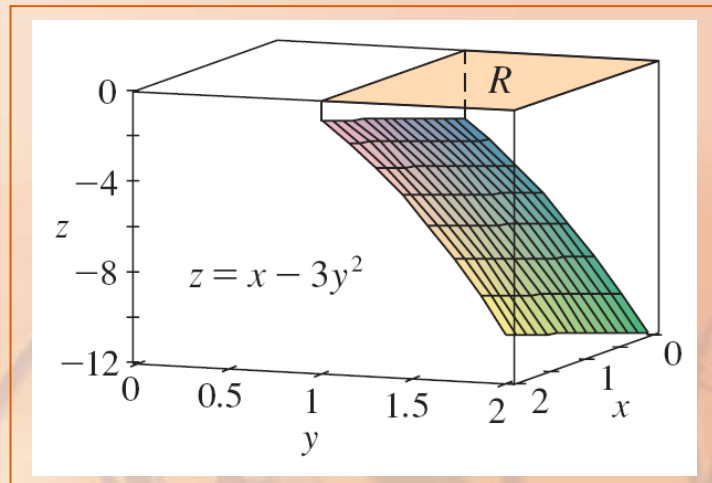
Nothing is wrong with that.

- The function f in the example is not a positive function.
- So, its integral doesn't represent a volume.

Example:

From the figure, we see that f is always negative on R .

- Thus, the value of the integral is the negative of the volume that lies above the graph of f and below R .



Example: Evaluate:

$$\iint_R (x + y) dA \text{ over } R = \{(x, y) | 1 \leq x \leq 3, -1 \leq y \leq 2\}$$

Example: Evaluate:

$$\iint_R (2x - y) dA \quad \text{over } R = \{(x, y) \mid -1 \leq x \leq 5, 2 \leq y \leq 4\}$$

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Example: Find the volume of the solid under the graph

$$f(x, y) = 1 + x^2 + y^2$$

over the rectangular $R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$

$\frac{5}{3}$ cubic units

Example: Find the volume of the solid under the graph

$$f(x, y) = x + 2y$$

over the rectangular $R = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 1\}$

5 cubic units

Example: Evaluate:

$$\iint_R 2xe^{x^2+y} dA \quad \text{over } R = \{(x, y) | 0 \leq x \leq 1, -1 \leq y \leq 1\}$$

$e^2 - e - e + e^{-1}$

Example: Evaluate:

$$\iint_R \frac{x}{y^2} e^{x/y} dA \quad \text{over } R = \{(x, y) | 0 \leq x \leq 1, 1 \leq y \leq 2\}$$

$$e - 2e^{1/2} + 1$$

Example: Evaluate:

$$\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} y(0.4e^{-x} e^{-0.4y}) dx \right] dy$$

$$\sim 2.5$$

AVERAGE VALUE: Review MA216

Recall that the average value of a function f of one variable defined on an interval $[a, b]$ is:

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

AVERAGE VALUE over Rectangular Regions

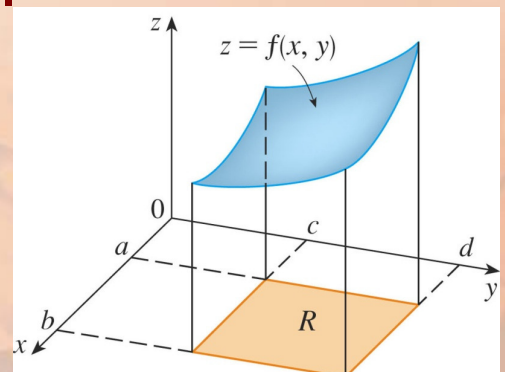
Similarly, we define the average value of a function f of two variables defined on a rectangle R to be:

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

where $A(R)$ is the area of R .

$$f_{ave} = \frac{1}{(b-a)(d-c)} \iint_R f(x, y) dA$$

$$R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$$



AVERAGE VALUE over Rectangular Regions

If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{ave} = \iint_R f(x, y) dA$$

says that:

- The box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f .

AVERAGE VALUE over Rectangular Regions

Suppose $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave} .

- Then, you can use them to fill in the valleys so that the region becomes completely flat.



Example: Find the average value of $f(x, y) = 4 - \frac{1}{2}x - \frac{1}{2}y$
over the rectangular $R = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 2\}$

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Example: Find the average value of $f(x, y) = x + 2y$
over the rectangular $R = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 1\}$

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PROPERTIES OF DOUBLE INTEGRALS

These properties are referred to as the linearity of the integral.

$$\iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$$

$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA$$

where c is a constant

PROPERTIES OF DOUBLE INTEGRALS

If $f(x, y) \geq g(x, y)$ for all (x, y) in R , then

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$$

Example: Applications

In a certain factory, output is given by the Cobb-Douglas production function

$$Q(K, L) = 50K^{3/5}L^{2/5}$$

Where K is the capital investment in units of 1,000,000 THB and L is the size of the labour force measured in worker-hours. Suppose that monthly capital investment varies between 10,000,000 THB and 12,000,000 THB, while monthly use of labour varies between 2,800 and 3,200 worker-hours. Find the average monthly output for the factory.

Ans: 5,181 unit