

## Exercise 5: Relation & Function

### 1 Part I

1. Let  $A = \{1, 2, 3\}$  and  $\mathbb{Z}$  be the set of all integers. Let  $\mathcal{P}(A)$  be the set of all subsets of the set  $A$ . Define a relation  $r$  from  $\mathcal{P}(A)$  to  $\mathbb{Z}$  as

$$r = \{(x, y) \in \mathcal{P}(A) \times \mathbb{Z} \mid y = \text{the number of elements in } x\}.$$

- (a) List all the elements in  $r$ .
  - (b) Is  $r$  a function? If so, find the domain, co-domain, and range of  $r$ .
2. Let  $X = \{1, 2, 3\}$  and  $Y = \{a, b, c, d\}$ . Define a function  $f : X \rightarrow Y$  by  $f = \{(1, a), (2, a), (3, c)\}$ .
- (a) Find the domain of  $f$ , co-domain of  $f$ , and range of  $f$ .
  - (b) What is the inverse image of  $a$ ?
  - (c) What is  $f(2)$ ?
  - (d) Draw the arrow diagram of  $f$ .

3. Define  $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  as follows:

$$H(x, y) = (2x + 1, \frac{1-y}{2}) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

- (a) Is  $H$  one-to-one? Prove or give a counterexample.
  - (b) Is  $H$  onto? Prove or give a counterexample.
  - (c) Is  $H$  bijective? If so, find  $H^{-1}$ , the inverse function of  $H$ .
4. Let  $f : (0, \infty) \rightarrow \mathbb{R}$  defined by

$$f = \begin{cases} |x|, & x \in (0, 1] \\ x^2, & x \in (1, 3] \\ 6 + |x|, & x \in (3, \infty). \end{cases}$$

- (a) Is  $f$  one-to-one? Prove or give a counterexample.
  - (b) Is  $f$  onto? Prove or give a counterexample.
  - (c) Is  $f$  bijective? If so, find the inverse function of  $f$ .
5. Find the largest sets  $D$  and  $S$  such that the function  $f : D \rightarrow S$  defined by  $f(x) = \frac{2}{x-1}$  is an onto function.
6. Let  $f : \mathbb{R} - \{1\} \rightarrow S$  be a *bijective* function defined by  $f(x) = \frac{x+1}{x-1}$ .
- (a) Determine the set  $S$ .
  - (b) Determine the inverse function  $f^{-1}$ .
  - (c) Compute  $f \circ f$  and  $f \circ f^{-1}$ .

7. Graph the function  $f : \mathbb{R} \rightarrow \mathbb{Z}^+ \cup \{0\}$ ,  $f(x) = \lfloor x \rfloor - \lceil x \rceil$  on the closed interval  $[-4, 4]$ .
8. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are functions and  $g \circ f : X \rightarrow Z$  is onto, must both  $f$  and  $g$  be onto? Prove or give a counterexample.
9. Let  $f$  and  $g$  be functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Find  $f \circ g$ ,  $g \circ f$ , and determine whether or not  $f \circ g = g \circ f$  for the given formulas for  $f$  and  $g$ . Compute  $(f \circ g)(2)$  and  $(g \circ f)(2)$ .

$$(a) f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$$

$$(b) f(x) = x^5, \quad g(x) = x^{1/5}.$$

10. Let  $f$  and  $g$  be functions defined by  $f(x) = \sqrt{2-x}, x \geq 2$  and

$$g = \begin{cases} x^2 - 1, & x \in (-\infty, -1] \\ \frac{1}{x}, & x \in (-1, \infty). \end{cases}$$

Find  $g \circ f$  and its domain.

## 2 Part II

1. Let  $X = \{1, 2, 3\}$  and  $Y = \{a, b, c, d\}$ . Define relations  
 $f : X \rightarrow Y$  by  $f = \{(1, a), (2, a), (3, c)\}$ ,  
 $g : X \rightarrow Y$  by  $g = \{(1, a), (3, c)\}$ , and  
 $h : X \rightarrow Y$  by  $h = \{(1, a), (2, a), (3, b), (3, c)\}$ .
- (a) Draw the arrow diagrams of  $f$ ,  $g$ , and  $h$ .
- (b) Show that  $f$  is a function, but  $g$  and  $h$  are not functions.
- (c) Find the domain of  $f$ , co-domain of  $f$ , and range of  $f$ .
- (d) What is the inverse image of  $a$  for the function  $f$ ?
- (e) What is  $f(3)$ ?
2. Let  $A = \{-1, 1, 2, 4\}$  and  $B = \{1, 2\}$  and define relations  $R$  and  $S$  from  $A$  to  $B$  as follows: For all  $(x, y) \in A \times B$ ,

$$x R y \Leftrightarrow |x| = |y|.$$

and

$$x S y \Leftrightarrow x - y \text{ is even.}$$

State explicitly the sets  $A \times B$ ,  $R$ ,  $S$ ,  $R \cup S$ , and  $R \cap S$ .

3. Let  $A = \{1, 2, 3\}$  and  $\mathbb{Z}$  be the set of all integers. Let  $\mathcal{P}(A)$  be the set of all subsets of the set  $A$ , and

$$X = \{x \in \mathcal{P}(A) \mid x \cap \{1\} \neq \emptyset\}.$$

Define a relation  $r$  from  $X$  to  $\mathbb{Z}$  as

$$r = \{(x, y) \in X \times \mathbb{Z} \mid y = \text{the number of elements in } x\}.$$

- (a) List all elements in  $X$ .

- (b) Draw an arrow diagram of  $r$  .  
 (c) Is  $r$  a function? If so, find the domain, co-domain, and range of  $r$  .

4. Define  $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  as follows:

$$H(x, y) = (x + 1, 2y - 3) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

- (a) Is  $H$  one-to-one? Prove or give a counterexample.  
 (b) Is  $H$  onto? Prove or give a counterexample.  
 (c) Is  $H$  bijective? If so, find  $H^{-1}$ , the inverse function of  $H$  .

5. Define functions  $f_1 : [0, 2) \rightarrow \mathbb{R}$  as

$$f_1(x) = x^2$$

and define  $f_2 : [2, \infty) \rightarrow \mathbb{R}$  as

$$f_2(x) = 3x - 2.$$

Let  $F : [0, \infty) \rightarrow [0, \infty)$  be a function defined by using  $f_1$  and  $f_2$ :

$F(x) = f_1(x)$ , for  $x \in [0, 2)$ , and  $F(x) = f_2(x)$ , for  $x \in [2, \infty)$ . That is,

$$F(x) = \begin{cases} x^2, & x \in [0, 2) \\ 3x - 2, & x \in [2, \infty). \end{cases}$$

- (a) Find the domain, co-domain, and range for each of the functions  $f_1$ ,  $f_2$ , and  $F$ .  
 (b) Construct the composite functions  $f_1 \circ f_2$ ,  $f_2 \circ f_1$ , and  $f_1 \circ F$  (if possible). Determine the domains and ranges for these composite functions.  
 (c) Are  $f_1$  and  $f_2$  injective? Explain.  
 (d) Is the function  $F$  bijective? If so, find the **inverse function** of  $F$ .  
 6. Let  $f$  and  $g$  be functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Find  $f \circ g$ ,  $g \circ f$ , and determine whether or not  $f \circ g = g \circ f$  for the given formulas for  $f$  and  $g$ . Compute  $(f \circ g)(2)$  and  $(g \circ f)(2)$ .

(a)  $f(x) = \frac{x}{\sqrt{x^2+1}}$ ,  $g(x) = x^3 + 1$ .

(b)  $f(x) = x^5$ ,  $g(x) = x^{1/5}$ .

7. Let  $f : \mathbb{R} - \{1\} \rightarrow \mathbb{R} - \{-2\}$  be a function defined by  $f(x) = \frac{2x+1}{1-x}$ .

- (a) Compute  $f \circ f$  and determine its domain.  
 (b) Determine whether  $f$  is bijective. If so, find the inverse function  $f^{-1}$  and  $f \circ f^{-1}$ .

### 3 Part III

1. Let  $X = \{2, 3, 4\}$  and  $Y = \{1, 5\}$ . Define relations

$f : X \rightarrow Y$  by  $f = \{(x, y) \in X \times Y \mid x > y\}$ ,

$g : X \rightarrow Y$  by  $g = \{(x, y) \in X \times Y \mid x = 2y\}$ , and

$h : X \rightarrow Y$  by  $h = \{(x, y) \in X \times Y \mid y = x^2\}$ .

- List all the elements of the Cartesian product  $X \times Y$ .
  - List all the elements of the relations  $f$ ,  $g$ , and  $h$ .
  - Determine which of the relations  $f, g, h$  is a function from  $X$  to  $Y$ . Explain your answer.
  - What is the inverse image of 1 under  $f$ ? What is the inverse image of 5 under  $f$ ?
2. Let  $A = \{1, 2, 3\}$  and let  $\mathcal{P}(A)$  be the set of all subsets of the set  $A$ . Define a relation  $r$  and  $s$  as

$$r = \{(x, y) \in A \times \mathcal{P}(A) \mid x = \text{the number of elements in } y\},$$

$$s = \{(u, v) \in \mathcal{P}(A) \times A \mid (v, u) \in r\}.$$

- Draw arrow diagrams of  $r$  and  $s$ .
  - Is  $r$  a function? If so, is it onto and/or one-to-one? Justify your answer.
  - Is  $s$  a function? If so, is it onto and/or one-to-one? Justify your answer.
3. Define  $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  as follows:

$$H(x, y) = (y, x - 2) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

- Is  $H$  one-to-one? Prove or give a counterexample.
  - Is  $H$  onto? Prove or give a counterexample.
  - Is  $H$  bijective? If so, find  $H^{-1}$ , the inverse function of  $H$ .
4. Define functions  $f$  and  $g$  as follows:

$f = \{(1, 10), (3, 30), (5, 50)\}$  and

$g = \{(10, k), (20, \ell), (30, m), (40, n), (50, t)\}$ .

- Determine the domain and range for each of functions  $f$  and  $g$ .
  - Find  $g \circ f$  and  $f \circ g$  (if possible) and the corresponding domain and range for each of them.
  - Find  $g \circ g^{-1}$  and  $g^{-1} \circ g$  (if possible) and their corresponding domains and ranges.
5. Define

$$g(x) = \frac{1}{\sqrt{x+1}} + 1 \quad \text{and} \quad F(x) = \begin{cases} x^2 - 1, & x \in [-3, 1) \\ 2 - x, & x \in [1, 4]. \end{cases}$$

- Find the domain and range for each of the functions  $g$  and  $F$ .
- Construct the composite functions  $F \circ g$ , and  $g \circ F$  (if possible). Determine the domains for these composite functions.
- Is the function  $F$  bijective? If so, find the **inverse function** of  $F$ . Justify your answer.