Partial-Fraction Decomposition

TU152: Fundamental Mathematics

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Partial-fraction decomposition

Partial-fraction decomposition is a useful tool for computing integration of functions in the form

$$\frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials with the degree (highest power of x) of $p(x)$ is smaller than the degree of $q(x)$. I.e. $\frac{p(x)}{q(x)}$ has to be a "proper" fraction.

- Partial-fraction decomposition only works for "proper" fractions. That is, if the denominator's degree is not larger than the numerator's degree (so you have, in effect, an "improper" polynomial fraction), then you first have to use long division to get the "mixed number" form of the rational expression. Then decompose the remaining fractional part.
- Partial fraction decomposition will be in the form of the sum of the *proper fractions* of whose denominators are polynomials of degree either 1 or "irreducible" polynomials of degree 2.

Definition

- A rational function $\frac{p(x)}{q(x)}$ is called a **proper fraction** if $p(x)$ and $q(x)$ are polynomials with the degree (highest power of x) of $p(x)$ is smaller than the degree of $q(x)$.
- A polynomial is **irreducible** if it cannot be expressed as the product of two or more such polynomials, each of them having a lower degree than the original one.

Partial-fraction decomposition

The process of partial fraction starts from factoring the denominator into polynomials of degree either 1 or "irreducible" polynomials of degree 2.

1. Factor of the form $ax + b$, $a \neq 0$. Partial fraction decomposition:
$$\frac{A}{ax+b}$$
2. Factor of the form $(ax + b)^n$, $a \neq 0$, $n \geq 2$. Partial fraction decomposition:
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots \frac{A_n}{(ax+b)^n}.$$
3. Factor of the form $ax^2 + bx + c$, $a \neq 0$. Partial fraction:
$$\frac{A}{ax+b}$$
4. Factor of the form $(ax^2 + bx + c)^n$, $a \neq 0$, $n \geq 2$. Partial fraction decomposition:
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \dots \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

The constants A, A_i, B_i above can be found by using one of the following techniques.

1. Compare the coefficients of the same degrees for x.
2. Substitute the certain values for x.

Example: Find the partial fraction decomposition for $\dfrac{6}{(x-1)(x+1)}$.

Example: Find the partial fractions decomposition for

$$\frac{2x^4 + 7x^2 + 4}{x^4 + x^2}.$$

Example (continued):

Example: Find the partial fraction decomposition for

$$\frac{3x^5 - x^4 + 7x^3 - 3x^2 + 3x - 2}{x^6 + 2x^4 + x^2}.$$

Example (continued):