5 Exponential and Logarithmic Functions

5.1 Exponential Functions (Revision)

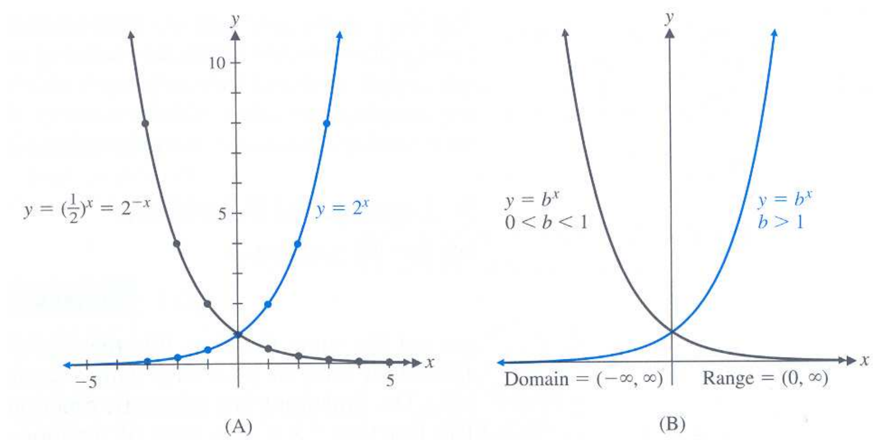
Exponential functions are used extensively in modelling and solving a while variety of real-world problems, including growth of money in compound interest; growth of populations of people, animals, and bacteria; radioactive decay; and learning associated with new technology e.g. computing and manufacturing technology.

5.1.1 General form:

$y = f(x) = b^x \qquad b > 0, b \neq 1$

Domain is ______________________ and range is ____________________.

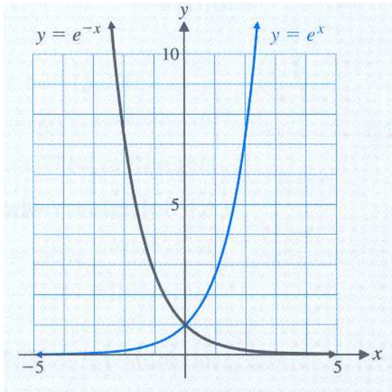
b is positive to avoid complex (imaginary) number.



- All graphs will pass through the point (0, 1).
- All graphs are continuous.
- The *x* axis is a horizontal asymptote.
- If $b > 1$, then b^x increases as *x* increases.
- If $0 < b < 1$, then b^x decreases as *x* decreases.

5.1.2 Base e Exponential Function

$e = 2.718\ 281\ 828\ 459 \ldots$



5.1.3 Exponential Properties

If n and m are positive integers,

$$b^{n/m} = \left(\sqrt[m]{b}\right)^n = \sqrt[m]{b^n}$$

where $\sqrt[m]{b}$ denotes the positive mth root

$$b^{-n} = \frac{1}{b^n}$$

$$b^0 = 1$$

$$b^x = b^y \quad \text{if and only if } x = y$$

$$b^m b^n = b^{m+n}$$

$$\left(b^n\right)^m = b^{mn}$$

$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$$

5.2 Logarithmic Functions (Revision)

Logarithmic function is inversed of exponential function.

5.2.1 General form:

$y = \log_b x$ *where* $x = b^y, b > 0, b \neq 1$

Domain is ______________________ and range is ____________________.

Note that $\log_b 1 = 0$. If the base b is 10, it is normally omitted. The natural logarithmic of x, $\ln(x)$ is $\log_e(x)$.

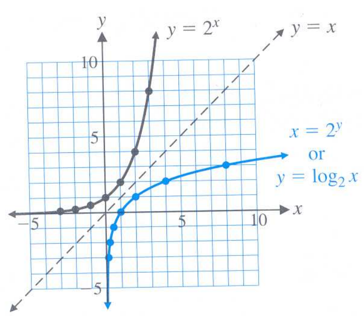


FIGURE 2

Exponential Function		Logarithmic Function	
x	$y = 2^x$	$x = 2^y$	y
−3	$\frac{1}{8}$	$\frac{1}{8}$	−3
−2	$\frac{1}{4}$	$\frac{1}{4}$	−2
−1	$\frac{1}{2}$	$\frac{1}{2}$	−1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

Ordered pairs reversed

5.2.2 Logarithmic Properties (b > 0 and $b \neq 1$, u and v are also positive numbers)

$\log_b 1 = 0$ and $\log_b b = 1$

$\log_b u = \log_b v$ if and only if $u = v$

$\log_b (uv) = \log_b u + \log_b v$

$\log_b \left(\frac{u}{v}\right) = \log_b u - \log_b v$

$\log_b u^r = r \log_b u$

$\log_b b^u = u$

$\log_b x = \frac{\ln x}{\ln b}$

5.3 Differentiation of Logarithmic and Exponential Functions

$\frac{d}{dx}\left(e^x\right)=e^x$	$\frac{d}{dx}\left(e^u\right)=e^u\frac{du}{dx}$
$\frac{d}{dx}(\ln x)=\frac{1}{x}$	$\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{du}{dx}$

Ex. 1: Find the derivative of the following functions.

(a) $y=2e^x$ Ans: $y'=2e^x$

(b) $y=2e^{3x}$ Ans: $y'=6e^{3x}$

(c) $y=ae^{-bt}$ Ans: $y'=-abe^{-bt}$

(d) $y=5e^{x^4+2x}$ Ans: $y'=5(4x^3+2)e^{x^4+2x}$

(e) $y=\frac{e^{-3x}}{x^2+1}$ Ans: $y'=e^{-3x}\left[\frac{-3x^2-2x-3}{\left(x^2+1\right)^2}\right]$

Ex.2: Show that the derivative of $f(x)=xe^{2x}$ is $f'(x)=(2x+1)e^{2x}$. In addition, find the largest and the smallest values of the function $f(x)=xe^{2x}$ on the interval $-1\le x\le 1$.

Ans: Minimum $f\left(-\frac{1}{2}\right)\approx -0.184$ and maximum $f(1)\approx 7.389$

Ex. 3: Find the derivative of the following functions.

(a) $y=5\ln x$ Ans: $y'=\frac{5}{x}$

(b) $y=\ln(5x)$ Ans: $y'=\frac{1}{x}$

(c) $y=x\ln x$ Ans: $y'=1+\ln x$

(d) $y = \dfrac{\ln \sqrt[3]{x^2}}{x^4}$ Ans: $y' = \dfrac{2}{3}\left[\dfrac{1-4\ln x}{x^5}\right]$

(e) $y = 3\ln(x^2 - 5x)$ Ans: $y' = \dfrac{3(2x-5)}{x^2-5x}$

(f) $y = (x + \ln x)^{\frac{3}{2}}$ Ans: $y' = \dfrac{3}{2}(x+\ln x)^{\frac{1}{2}}\left(1+\dfrac{1}{x}\right)$

(g) $y = \log_b x$ Ans: $y' = \dfrac{1}{(\ln b)x}$

(h) $y = \log_b u$ (u is a function of x) Ans: $y' = \dfrac{1}{(\ln b)u}\cdot\dfrac{du}{dx}$

Generalised Derivative of Logarithmic Functions

$$\boxed{\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u}\cdot\frac{du}{dx}}$$

Ex. 4: Show that the derivative of $f(x) = x - \ln\sqrt{x}$ is $f'(x) = 1 - \dfrac{1}{2x}$. In addition, show that an equation for the tangent line to the graph of $f(x) = x - \ln\sqrt{x}$ at the point where $x = 1$ is $y = 0.5x + 0.5$.

Derivative of Exponential Functions to the Base 'a' (a is a real number)

We can use the formula for the derivatives of e^u to differentiate a more general exponential function a^u. Replacing a with its equivalent from $e^{\ln a}$. $\left[a = e^{\ln a}\right]$

$$\begin{aligned}\frac{d(a^u)}{dx} &= \frac{d}{dx}\left[\left(e^{\ln a}\right)^u\right]\\ &= \frac{d}{dx}\left(e^{u\ln a}\right)\\ &= e^{u\ln a}\frac{d}{dx}(u\ln a)\\ &= e^{u\ln a}\left(\frac{du}{dx}\right)(\ln a)\end{aligned}$$

$$\boxed{\frac{d(a^u)}{dx} = a^u \ln a\left(\frac{du}{dx}\right)}$$

Ex. 4: Determine the derivative of the following functions.

(a) $y = a^x$. Ans: $y' = a^x \ln a$

(b) $y = 20^x$ Ans: $y' = 20^x \ln 20$

(c) $y = x2^{3x}$ Ans: $y' = 2^{3x}(1 + 3x\ln 2)$

(d) $y = \alpha 10^{-x} + \beta 10^{-x^2}$ Ans: $y' = -\ln 10\left(\alpha 10^{-x} + 2\beta x 10^{-x^2}\right)$

5.4 Logarithmic Differentiation

This technique is often useful to simplify the differentiation of $y = f(x)$ when $f(x)$ involves products, quotients or powers.

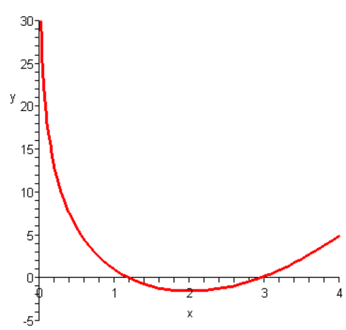
Ex. 5: Show that the derivative of $y = 2^x$ is $y' = (\ln 2)2^x$.

Ex. 6: Show that the derivative of $y = (x+1)^2(x+2)$ is $y' = (x+1)(3x+5)$.

Ex. 7: Show that the derivative of $y = x^{2x+1}$ is $y' = x^{2x+1}\left[2\ln x + \frac{2x+1}{x}\right]$.

5.5 Curve Sketching

Ex. 8: $f(x) = x^2 - 8\ln x$ intercepts the x axis at $x \approx 1.2$ and $x \approx 2.9$. Sketch the graph of $f(x) = x^2 - 8\ln x$.



Ans: Domain is positive real number.

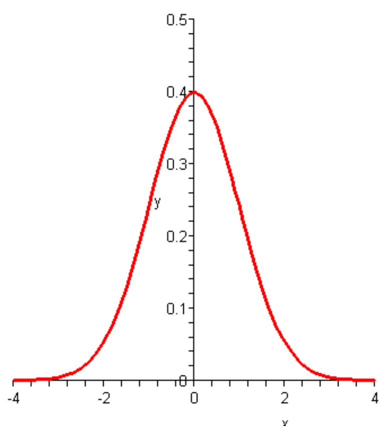
$$f'(x) = \frac{2x^2 - 8}{x}$$

$f(x)$ is decreasing between (0,2) and is increasing between $(2, \infty)$. The minimum point is (2,-1.5). The function is always concave up and there is no inflection point. $x = 0$ is the vertical asymptote and there is no horizontal asymptote.

Ex. 9: Determine where the function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is increasing and decreasing, and where its graph is concave up and concave down. Find the relative extrema and inflection points and draw the graph.



Ans: $f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$,

$f''(x) = \frac{-x}{\sqrt{2\pi}}\left(x^2 - 1\right)e^{-\frac{x^2}{2}}$ (0, 0.4) is the relative maximum. (1, 0.24) and (-1, 0.24) are inflection points. The function is increasing for negative x and decreasing for positive x. The function is concave up where $x < -1$ and $x > 1$; and concave down where $-1 < x < 1$. There is no x intercept. y = 0 is the horizontal asymptote.