EE481 : Homework 2 Solution

29/30

Assignment 2 (due 14/09/2018)

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Suppose that demand is given by $P = 300 - Q$ and marginal cost equals 10. Firms are Cournot competitors and play a supergame. The collusive agreement being considered is for each to produce one-fourth of the monopoly output (there are 4 firms in this industry.

10/10

a) What is the critical discount factor to sustain collusion using grim punishment strategies if detection of deviation requires 2 periods?

PV from colluding $\rightarrow V_i^* = \pi_i^*\left(\frac{1}{1-\delta}\right)$; $0 \le \delta \le 1$

PV from cheating $\rightarrow$
$$V_i^r = \pi_i^r + \delta\pi_i^r + \delta^2\pi_i^c + \delta^3\pi_i^c + \dots + \delta^\infty\pi_i^c$$
$$= \pi_i^r + \delta\pi_i^r + \delta^2\pi_i^c(1+\delta+\dots+\delta^\infty)$$
$$= \pi_i^r + \delta\pi_i^r + \frac{\delta^2\pi_i^c}{1-\delta}$$

In order for firm i to not cheat, $V_i^* > V_i^r$.

$$\frac{\pi_i^*}{1-\delta} > \pi_i^r + \delta\pi_i^r + \frac{\delta^2\pi_i^c}{1-\delta} \quad ; \text{ multiply both sides by } 1-\delta.$$
$$\pi_i^* > \pi_i^r(1-\delta) + \delta\pi_i^r(1-\delta) + \delta^2\pi_i^c$$
$$\pi_i^* > \pi_i^r - \cancel{\delta\pi_i^r} + \cancel{\delta\pi_i^r} - \delta^2\pi_i^r + \delta^2\pi_i^c$$
$$\pi_i^* - \pi_i^r > -\delta^2(\pi_i^r - \pi_i^c)$$
$$\delta^2(\pi_i^r - \pi_i^c) > \pi_i^r - \pi_i^*$$
$$\delta^2 > \frac{\pi_i^r - \pi_i^*}{\pi_i^r - \pi_i^c}$$
$$\delta > \sqrt{\frac{\pi_i^r - \pi_i^*}{\pi_i^r - \pi_i^c}}$$

Notations

π_i^* = profit from successful collusion.

π_i^r = profit from cheating given that all other firms still collude

π_i^c = profit when firms do not cooperate.

I. Find profit under perfect cartel (π_i^*)

$$\pi_i^* = \frac{\pi^{monopoly}}{4} = \frac{21{,}025}{4} = 5{,}256.25$$

$$\pi^{mono} = TR - TC$$
$$= P \cdot Q - mc \cdot Q$$
$$= 300Q - Q^2 - 10Q$$
$$= 290Q - Q^2$$

FOC: $\frac{\partial\pi^{mono}}{\partial Q} = 290 - 2Q = 0$

$Q^* = 145$

$P^* = 300 - 145 = 155$

$$\pi^{mono} = 290(145) - (145)^2 = 21{,}025$$

II. Find profit of a firm under non-cooperation. (π_i^c)

$$\pi_i^c = TR - TC$$
$$= [P(Q) \cdot q_i^c] - [mc \cdot q_i^c]$$
$$= [(300-Q)q_i^c] - [10q_i^c]$$
$$= 300q_i^c - Qq_i^c - 10q_i^c$$
$$= 290q_i^c - Qq_i^c$$
$$= 290q_i^c - (q_i^c + 3q_j^c)q_i^c$$
$$= 290q_i^c - (q_i^c)^2 - 3q_j^cq_i^c$$

FOC: $\frac{\partial\pi_i^c}{\partial q_i^c} = 290 - 2q_i^c - 3q_j^c = 0$

$q_i^c = 145 - \frac{3}{2}q_j^c$: Best response function of firm i

By symmetry $q_j^c = 145 - \frac{3}{2}q_i^c$: Best response function of firm j

Notations

Q = Total quantity.

q_j^c = Quantity of the other firm

Find Cournot equilibrium.

$$q_i^c = 145 - \tfrac{3}{2}[145 - \tfrac{3}{2}q_i^c]$$
$$q_i^c = 145 - 217.5 + 2.25q_i^c$$
$$1.25q_i^c = 72.5$$
$$q_i^c = 58$$

By the similar calculation, $q_j^c = 58$.

So, total quantity $(Q) = q_i^c + 3q_j^c$
$$= 58 + 3(58)$$
$$= 232.$$

$$P = 300 - Q = 300 - 232 = 68$$

Profit π_i^c

$$\pi_i^c = TR - TC$$
$$= P \cdot q_i^c - mc \cdot q_i^c$$
$$= (P - mc)q_i^c$$
$$= (68-10)(58)$$
$$\pi_i^c = 3{,}364.$$

III. Find profit from cheating (π_i^r)

$$\pi_i^r = TR - TC$$
$$= [P(Q)\cdot q_i^r] - [mc\cdot q_i^r]$$
$$= [(300-Q)q_i^r] - [10q_i^r]$$
$$= [300q_i^r - Qq_i^r] - [10q_i^r]$$
$$= [300q_i^r - (q_i^r + 3q^{cartel})q_i^r] - [10q_i^r]$$
$$= 300q_i^r - (q_i^r)^2 - 3q^{carter}q_i^r - 10q_i^r$$
$$= 290q_i^r - (q_i^r)^2 - 3q^{cartel}q_i^r$$
$$= 290q_i^r - (q_i^r)^2 - 3(36.25)q_i^r$$
$$= 290q_i^r - (q_i^r)^2 - 108.75q_i^r$$
$$= 181.25q_i^r - (q_i^r)^2$$

FOC: $\frac{\partial \pi_i^r}{\partial q_i^r} = 181.25 - 2q_i^r = 0$

$$q_i^r = 90.625$$

So, total quantity $(Q) = q_i^r + 3q^{cartel}$
$$= 90.625 + 3(36.25)$$
$$= 90.625 + 108.75$$
$$= 199.375$$
$$P = 300 - Q$$
$$= 300 - 199.375$$
$$= 100.625$$

$$\pi_i^r = TR - TC$$
$$= (P - MC)q_i^r$$
$$= (100.625 - 10)(90.625)$$
$$\pi_i^r = 8{,}212.890625 \approx 8{,}212.89$$ ✓

Notations

q_i^r = Quantity of the cheating firm.

$q^{cartel} = \frac{145}{4} = 36.25$.

IV. Find the critical discount factor (δ).

As derived at the beginning,

$$\delta > \sqrt{\frac{\pi_i^r - \pi_i^*}{\pi_i^r - \pi_i^c}}$$
$$\delta > \sqrt{\frac{8{,}212.89 - 5256.25}{8{,}212.89 - 3{,}364}}$$
$$\delta > \sqrt{0.60975}$$
$$\delta > 0.78086 \approx 0.78$$ ✓

Ans In order to sustain the collusion, the discount factor of firms must be greater than the critical discount factor, $\delta > 0.78$.

b) Do you think the value of the critical discount factor will be higher or lower if detection of deviation requires 3 periods? Explain.

9.5/10

PV from colluding → $V_i^* = \pi_i^*\left(\frac{1}{1-\delta}\right)$; $0 \leq \delta \leq 1$

PV from cheating → $V_i^r = \pi_i^r + \delta\pi_i^r + \delta^2\pi_i^r + \delta^3\pi_i^c + \ldots + \delta^\infty\pi_i^c$
$$= \pi_i^r + \delta\pi_i^r(1+\delta) + \delta^3\pi_i^c(1+\delta+\ldots+\delta^\infty)$$
$$= \pi_i^r + \delta\pi_i^r(1+\delta) + \frac{\delta^3\pi_i^c}{1-\delta}$$

In order to sustain collusion, $V_i^* > V_i^r$

$$\pi_i^*\left(\frac{1}{1-\delta}\right) > \pi_i^r + \delta\pi_i^r(1+\delta) + \frac{\delta^3\pi_i^c}{1-\delta}$$; multiply both sides by $1-\delta$.

$$\pi_i^* > (1-\delta)\pi_i^r + \delta\pi_i^r(1-\delta)(1+\delta) + \delta^3\pi_i^c$$
$$\pi_i^* > \pi_i^r - \cancel{\delta\pi_i^r} + \cancel{\delta\pi_i^r} - \delta^3\pi_i^r + \delta^3\pi_i^c$$
$$\pi_i^* - \pi_i^r > -\delta^3(\pi_i^r - \pi_i^c)$$
$$\delta^3(\pi_i^r - \pi_i^c) > \pi_i^r - \pi_i^*$$
$$\delta^3 > \frac{\pi_i^r - \pi_i^*}{\pi_i^r - \pi_i^c}$$
$$\delta > \sqrt[3]{\frac{\pi_i^r - \pi_i^*}{\pi_i^r - \pi_i^c}}$$

So, $$\delta > \sqrt[3]{\frac{8{,}212.89 - 5256.25}{8{,}212.89 - 3{,}364}}$$
$$\delta > \sqrt[3]{0.60975}$$
$$\delta > 0.8479767 \approx 0.85$$ ← not required to calculate but gets +0.5 bonus

Ans The value of the critical factor is higher when the detection of deviation requires 3 periods.

Should also explain why. But for now, it's OK.

C) Do you think the value of the critical discount factor will be higher or lower if the number of firms increases to 5? Explain.

9.5/10

If there are 5 firms in the industry,

I. Profit under perfect cartel of each firm (π_i^*) $= \frac{21{,}025}{5} = 4{,}205$.

II. Profit when there is no cooperation (π_i^c)

$$\pi_i^c = TR - TC$$
$$= [P(Q)\cdot q_i^c] - [mc\cdot q_i^c]$$
$$= [(300-Q)q_i^c] - [10q_i^c]$$
$$= [300q_i^c - Qq_i^c] - [10q_i^c]$$
$$= [300q_i^c - (q_i^c + 4q_j^c)q_i^c] - [10q_i^c]$$
$$= 290q_i^c - (q_i^c)^2 - 4q_j^c q_i^c$$

FOC: $\frac{\partial \pi_i^c}{\partial q_i^c} = 290 - 2q_i^c - 4q_j^c = 0$

$q_i^c = 145 - 2q_j^c$: Best response function of firm i.

By symmetry $q_j^c = 145 - 2q_i^c$: Best response function of firm j.

Cournot equilibrium

$$q_i^c = 145 - 2(145 - 2q_i^c)$$
$$q_i^c = -145 + 4q_i^c$$
$$3q_i^c = 145$$
$$q_i^c = 48.33$$

By the similar calculation, $q_j^c = 48.33$

Total quantity $(Q) = q_i^c + 4q_j^c$

$= 48.33 + 4(48.33)$

$= 241.6666 \approx 241.67$.

$P = 300 - 241.67$

$= 58.33$

$$\pi_i^c = TR - TC$$
$$= (P - mc)q_i^c$$
$$= (58.33 - 10)(48.33)$$
$$\pi_i^c = 2{,}335.7889 \approx 2{,}335.79.$$

III. Profit from cheating (π_i^r)

$$\pi_i^r = TR - TC$$
$$= [P(Q)\cdot q_i^r] - [mc\cdot q_i^r]$$
$$= [(300-Q)q_i^r] - [10q_i^r]$$
$$= [300q_i^r - Qq_i^r] - [10q_i^r]$$
$$= [300q_i^r - (q_i^r + 4q^{cartel})q_i^r] - [10q_i^r]$$
$$= [300q_i^r - (q_i^r)^2 - 4q^{cartel}q_i^r] - [10q_i^r]$$
$$= 290q_i^r - (q_i^r)^2 - 4(29)q_i^r$$
$$= 290q_i^r - (q_i^r)^2 - 116q_i^r$$
$$= 174q_i^r - (q_i^r)^2$$

FOC: $\frac{\partial \pi_i^r}{\partial q_i^r} = 174 - 2q_i^r = 0$

$q_i^r = 87$

$$\pi_i^r = 174q_i^r - (q_i^r)^2$$
$$= 174(87) - (87)^2$$
$$\pi_i^r = 7{,}569$$

IV. Find the critical discount factor (δ)

- If the detection of the deviation requires 2 periods,

$$\delta > \sqrt{\frac{\pi_i^r - \pi_i^*}{\pi_i^r - \pi_i^c}}$$
$$\delta > \sqrt{\frac{7{,}569 - 4{,}205}{7{,}569 - 2{,}335.79}}$$
$$\delta > \sqrt{0.6428}$$
$$\delta > 0.80176 \approx 0.801$$

- If the detection requires 3 periods,

$$\delta > \sqrt[3]{\frac{\pi_i^r - \pi_i^*}{\pi_i^r - \pi_i^c}}$$
$$\delta > \sqrt[3]{0.6428}$$
$$\delta > 0.863$$

not required but gets 0.5

<u>Ans</u> By comparing the calculations in questions a), b) and c), it can be observed that the increase in the number of firms also increase the critical discount factor.
In other words, the critical discount factor is higher when there are more firms in the industry.

Should also explain why? But for now, it's OK.

How to give explanation for part b) and c)?

b) Since it takes more time for firms to detect cheating, the cheating firm would be able to reap more profit. This should make cheating become more attractive. The critical discount factor (minimum value of discount factor required for a firm not to cheat) should become higher.

c) Since there are now more firms sharing the cartel (joint-monopoly profit), the collusion profit (π_i^*) should be lower. However, each cheating firm would get relatively more profit when they cheat (π^r). To sum up the effects

$$\delta \uparrow = \sqrt{\left(\frac{\pi^r \uparrow - \pi^* \downarrow}{\pi^r - \pi^c}\right) \uparrow}$$

This should making cheating "more" attractive. Therefore, the critical discount factor increases.