The Logic of Compound Statements: II

TU152: Fundamental Mathematics

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Outline

1. Statements with Multiple Quantifiers

2. Arguments with Quantified Statements

Multiple Quantifiers

Interpreting Statements with Two Different Quantifiers

- To determine the truth of a statement of the form

$$\boxed{\forall x \in D\ \exists y \in E, P(x, y)} \text{ or}$$

$$\forall x \in D, \exists y \in E \ \text{ such that } \ P(x, y),$$

  we have to show that:
  for whatever element x in D is chosen, we must find an element y in E that "works" (i.e. $P(x, y)$ is true) for that particular x.

- To determine the truth of a statement of the form

$$\boxed{\exists x \in D\ \forall y \in E,\ P(x, y)} \text{ or}$$

$$\exists x \in D \ \text{ such that } \ \forall y \in E,\ P(x, y),$$

  we have to *find one particular x in D that will "work" (i.e. $P(x, y)$ is true) for every element y in E.*

Multiple Quantifiers

Example:

(i) Let $P(x, y)$ denote the statement "$x + y = y + x$." Determine the truth value of the statement

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}, \quad P(x, y).$$

(ii) Let $Q(x, y)$ denote the statement "$x + y = 0$." Determine the truth value of the statement

$$\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}, \quad Q(x, y)$$

Answer:

Multiple Quantifiers

Example: Let $P(x, y)$ denote the statement "$xy = 1$," where the domain of x is the set of positive integers $\mathbb{Z}^+$ and the domain of y is the set of all real numbers $\mathbb{R}$.
Determine the truth value of the following statements.

(i) For every positive integer x and for every real number y, "$xy = 1$."

$$\forall x \in \mathbb{Z}^+ \ \forall y \in \mathbb{R}, P(x, y)$$

(ii) For every positive integer x there is a real number y such that $xy = 1$.

$$\forall x \in \mathbb{Z}^+ \ \exists y \in \mathbb{R}, P(x, y)$$

(iii) There exists a real number y such that, for every positive integer x, $xy = 1$.

$$\exists y \in \mathbb{R} \ \forall x \in \mathbb{Z}^+, P(x, y)$$

Answer:

Negations of Multiply-Quantified Statements

Recall that, if we let $Q(x)$ be a predicate and D be the domain of x

$$\boxed{\sim \Big(\forall x, Q(x)\Big) \equiv \exists x, \sim Q(x)}$$

and

$$\boxed{\sim \Big(\exists x, Q(x)\Big) \equiv \forall x, \sim Q(x).}$$

Negations of Multiply-Quantified Statements

- $$\sim \Big(\forall x \in D,\ \exists y \in E, P(x,y)\Big) \equiv \exists x \in D, \forall y \in E, \sim P(x,y)$$
- $$\sim \Big(\exists x \in D,\ \forall y \in E, P(x,y)\Big) \equiv \forall x \in D, \exists y \in E, \sim P(x,y)$$

In general, if we explicitly define D to be the domain of x and E to be the domain of y, then we can also write:

$$\boxed{\sim \Big(\forall x \exists y, P(x,y)\Big) \equiv \exists x \forall y, \sim P(x,y)}$$

and

$$\boxed{\sim \Big(\exists x \forall y, P(x,y)\Big) \equiv \forall x \exists y, \sim P(x,y)}$$

Negations of Multiply-Quantified Statements

Show that

$$\boxed{\sim \left(\forall x \exists y, P(x,y)\right) \equiv \exists x \forall y, \sim P(x,y).}$$

From $\sim \left(\forall x, Q(x)\right) \equiv \exists x, \sim Q(x)$, and $\sim \left(\exists x, Q(x)\right) \equiv \forall x, \sim Q(x)$

$$\begin{aligned} \sim \left(\forall x \exists y, P(x,y)\right) &\equiv \sim \left(\forall x \left(\exists y, P(x,y)\right)\right) \\ &\equiv \exists x, \sim \left(\exists y, P(x,y)\right) \\ &\equiv \exists x \forall y, \sim P(x,y) \end{aligned}$$

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Show that

$$\boxed{\sim \left(\exists x \forall y, P(x,y)\right) \equiv \forall x \exists y, \sim P(x,y).}$$

From $\sim \left(\forall x, Q(x)\right) \equiv \exists x, \sim Q(x)$, and $\sim \left(\exists x, Q(x)\right) \equiv \forall x, \sim Q(x)$

$$\begin{aligned} \sim \left(\exists x \forall y, P(x,y)\right) &\equiv \sim \left(\exists x \left(\forall y, P(x,y)\right)\right) \\ &\equiv \end{aligned}$$

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Negations of Multiply-Quantified Statements

Example: Let D_x, D_y, and D_z be the domains for x, y, and z, respectively. Express the negations of the statement:

$$\forall x \exists y \forall z, T(x, y, z)$$

so that all negation symbols $\sim$ precede predicates.

Answer:

$$\begin{aligned} \sim \Big(\forall x \exists y \forall z, T(x, y, z)\Big) &\equiv \; \sim \forall x \Big(\exists y \forall z, T(x, y, z)\Big) \\ &\equiv \end{aligned}$$

Negations of Multiply-Quantified Statements

Example: Let D_x, D_y be the domains for x, y, respectively. Express the negations of the statement:

$$\forall x \exists y, P(x,y) \ \vee \ \forall x \exists y, Q(x,y)$$

so that all negation symbols $\sim$ precede predicates.

Answer:

$$\sim \left(\forall x \exists y, P(x,y) \ \vee \ \forall x \exists y, Q(x,y) \right) \quad \equiv$$

$$\equiv$$

Order of Quantifiers

Example: Let $R(x, y)$ be the predicate "x understands y," where the domain of x is the set of students in this TU152 class and the domain of y is the set of examples in these lecture notes. Write the following statements using the quantifiers $\forall$, $\exists$, and the predicate $R(x, y)$.

(1) There exists a student in this class who understands every example in these lecture notes.
Answer:

(2) For every example in these lecture notes there is at least one student in the class who understands that particular example (it is possible that different students understand different examples).
Answer:

(3) Every student in this class understands at least one example in these notes.
Answer:

(4) There is an example in these notes that every student in this class understands.
Answer:

Notice

$$\exists x \forall y, R(x, y) \not\equiv \forall y \exists x, R(x, y) \quad \text{and} \quad \forall x \exists y, R(x, y) \not\equiv \exists y \forall x, R(x, y)$$

Order of Quantifiers

Example:

Arguments with Quantified Statements

The rule of universal instantiation

"If some property is true of everything in a set, then it is true of any particular thing in the set."

Example:

All human beings are mortal.
John is a human being.
∴ John is mortal.

- Universal instantiation is the fundamental tool of deductive reasoning.
- Mathematical formulas, definitions, and theorems are like general templates that are used over and over in a wide variety of particular situations.
- A given theorem says that such and such is true for all things of a certain type. If, in a given situation, you have a particular object of that type, then by universal instantiation, you conclude that such and such is true for that particular object.

Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called universal modus ponens.

Universal Modus Ponens

Formal Version	**Informal Version**
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore Q(a)$.	$\therefore$ a makes $Q(x)$ true.

Universal Modus Ponens

Example: Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

k is a particular integer that is even.

$\therefore$ k^2 is even.

Answer:

Universal Modus Tollens

Universal modus tollens is the main concept of proof by contradiction, which is one of the most important methods of mathematical argument.

Universal Modus Tollens

Formal Version	**Informal Version**
$\forall x, P(x) \rightarrow Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$ for a particular a.	a does not make $Q(x)$ true.
$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Universal Modus Tollens

Example: Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All lawyers went to law schools.
Tom didn't go to a law school.
$\therefore$ Tom is not a lawyer.

Answer:

Universal Transitivity

Universal Transitivity

Formal Version	Informal Version
$\forall x, P(x) \rightarrow Q(x).$	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\forall x, Q(x) \rightarrow R(x).$	If x makes $Q(x)$ true, then x makes $R(x)$ true.
$\therefore \forall x, P(x) \rightarrow R(x).$	$\therefore$ If x makes $P(x)$ true, then x makes $R(x)$ true.

Validity of Arguments with Quantified Statements

Definition

- To say that an argument form is **valid** means the following:

  No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

- An argument is called **valid** if, and only if, its form is valid.

Using Diagrams to Show Validity& Invalidity

Example: Use a diagram to show the validity of the following argument:

All lawyers went to law schools.
Tom didn't go to a law school.
$\therefore$ Tom is not a lawyer.

Using Diagrams to Show Validity& Invalidity

Example: Use a diagram to show the invalidity of the following argument:

All lawyers went to law schools.

Tom went to a law school.

$\therefore$ Tom is a lawyer.

Using Diagrams to Show Validity& Invalidity

Example: Use a diagram to show the invalidity of the following argument:

All freshmen must take TU152.

Jane takes TU152.

$\therefore$ Jane is a freshman.

Converse & Inverse Errors (Quantified Form)

The following arguments are invalid.

Converse Error (Quantified Form)

Formal Version	Informal Version
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$.	$\therefore$ a makes $P(x)$ true.

Inverse Error (Quantified Form)

Formal Version	Informal Version
$\forall x, P(x) \to Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$ for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$.	$\therefore$ a does not make $Q(x)$ true.

Exercises: Rewrite the following arguments using quantifiers, variables, and predicate symbols. Determine if these arguments are valid. Explain your answers.

(a)

All human beings are mortal.

Buster is mortal.

∴ **Buster** is a human being.

(b)

Any sum of two rational numbers is rational.

The sum $r + s$ is rational.

∴ The numbers r and s are both rational.

(c)

All freshmen must take TU152.

Jane is a freshman.

∴ Jane must take TU152.

(d)

All healthy people eat an apple a day

Jane eats an apple a day.

∴ Jane is a healthy person.