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Question 1:

Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?
1.2 Is the production function increasing with respect to K and L?
1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.
1.4 Use the Hessian matrix. Proof that the production function is concave.

1.1 $Q = f(K,L) = 18 \times [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$

$K_0 L_0 \; ; Q_0 = 18 \times [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$

$tK_0, tL_0 \; ; Q = 18 \times [0.2(tK_0)^{-0.4} + 0.8(tL_0)^{-0.4}]^{-2.5}$

$Q = (t^{-0.4})^{-2.5} \times 18 \times [(0.2K_0^{-4} + 0.8L_0^{-0.4}]^{-2.5}$

$Q = t^{1} \times 18 \times [(0.2K_0^{-4} + 0.8L_0^{-0.4}]^{-2.5}$

$\therefore Q = t^{1} \times Q_0$

This production function is constant return to scale because the degree of HoMogeneity is equal to 1

1.2 $\dfrac{\partial Q}{\partial K} = -45[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (-0.08K^{-1.4})$

$= 3.6[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (K^{-1.4})$

$= \dfrac{1.44K^{1.4} + 2.88L^{1.4}}{K^{1.4}} > 0$

if $K \uparrow$ 1 unit Q will increase by $\dfrac{1.44K^{1.4} + 2.88L^{1.4}}{K^{1.4}}$ unit

$\dfrac{\partial Q}{\partial L} = -45[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (-0.32L^{-1.4})$

$= 14.4[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (L^{-1.4})$

$= \dfrac{2.88K^{1.4} + 11.52L^{1.4}}{L^{1.4}} > 0$

if $L \uparrow$ 1 unit Q will increase by $\dfrac{2.88K^{1.4} + 11.52L^{1.4}}{L^{1.4}}$ unit

1.3 Let $f(k,L) = 18[0.2k^{-0.4} + 0.8L^{-0.4}]^{-2.5} = 0$

use implicit function rules $\frac{dk}{dL} = \frac{-F_L}{F_k}$

$$F_L = \frac{\partial F}{\partial L} = -45[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (-0.32L^{-1.4})$$
$$= 14.4[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (L^{-1.4})$$
$$F_k = \frac{\partial F}{\partial k} = -45[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (-0.08k^{-1.4})$$
$$= 3.6[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (k^{-1.4})$$

$$MRTS = \frac{dk}{dL} = \frac{-F_L}{F_k} = -\frac{\frac{\partial F}{\partial L}}{\frac{\partial F}{\partial k}} = \frac{-14.4\cancel{[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}} \cdot (L^{-1.4})}{3.6\cancel{[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}} \cdot (k^{-1.4})}$$
$$MRTS = -4 \cdot \frac{k^{1.4}}{L^{1.4}} = -4\left(\frac{K}{L}\right)^{1.4}$$

1.4 $f(k,L) = 18[0.2k^{-0.4} + 0.8L^{-0.4}]^{-2.5}$

$$H = \begin{bmatrix} f_{kk} & f_{kL} \\ f_{Lk} & f_{LL} \end{bmatrix}$$

$$f_k = \frac{\partial f}{\partial k} = -45[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5} \cdot (-0.08k^{-1.4})$$
$$= 3.6k^{-1.4}[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}$$
$$f_L = \frac{\partial f}{\partial L} = 14.4L^{-1.4}[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}$$

$$f_{kk} = \frac{\partial f_k}{\partial k} = (3.6k^{-1.4})(-3.5[0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-0.08k^{-1.4}) + [0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04k^{-2.4})$$
$$= (1.008k^{-2.8})[0.2^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04k^{-2.4})$$
$$f_{kL} = \frac{\partial f_k}{\partial L} = (3.6k^{-1.4})(3.5)[0.2k^{-0.4} + 0.8L^{-0.4}]^{-4.5}(-0.32L^{-1.4}) + [0.2k^{-0.4} + 0.8L^{-0.4}]^{-3.5}(0)$$
$$= (4.032k^{-1.4}L^{-1.4})[0.2k^{-0.4} + 0.8L^{-0.4}]^{-4.5}$$

$$f_{Lu} : \frac{\partial f_L}{\partial u} = (14.4L^{-1.4})(-3.5)[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5}(-0.08u^{-1.4}) + [0.2u^{-0.4} + 0.8L^{-0.4}]^{-3.5}(0)$$

$$= (4.032u^{-1.4}L^{-1.4})[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5}$$

$$f_{LL} : \frac{\partial u_L}{\partial L} = (14.4L^{-1.4})(-3.5)[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.w}(-0.32L^{-1.4}) + [0.2u^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-20.16L^{-2.4})$$

$$= (16.128L^{-2.8})[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2u^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-20.16L^{-2.4})$$

$$|H_1| = |(1.008u^{-2.8})[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2u^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04u^{-2.4})|$$

$$= (1.008u^{-2.8})[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2u^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04u^{-2.4})$$

$$|H_2| = \Big[\{(1.008u^{-2.8})[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2u^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04u^{-2.4})\} \times \{(16.128L^{-2.8})[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2u^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-20.16L^{-2.4})\}\Big]$$

$$- \Big[\{(4.032u^{-1.4}L^{-1.4})(0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5}\} \times \{(4.032u^{-1.4}L^{-1.4})[0.2u^{-0.4} + 0.8L^{-0.4}]^{-4.5}\}\Big]$$

$|H_1| < 0$; $\forall u, \forall L$ $\quad \therefore$ H is negative definite; $d^2 a < 0$

$|H_2| > 0$; $\forall u, \forall L$ $\quad$ So production is concave

Question 2: Define f(x,y) for all (x,y) by

$$f\,(x,y) \;=\; e^{\,x\,+y} + \,e^{\,x\,-y} - \tfrac{3}{2}\,x \,-\, \tfrac{1}{2}y$$

2.1 Derive the Hessian matrix of $f\,(x,y)$.

2.2 Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).

2.3 Find the *global* extrema of f(x,y). What type of extrema is it?

2.1) $H = \begin{bmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{bmatrix}$

$F_x = e^{x+y} + e^{x-y} - \frac{3}{2}$

$F_y = e^{x+y} - e^{x-y} - \frac{1}{2}$

$H = \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$ ※

2.2) $|H_1| = e^{x+y} + e^{x-y} > 0$ for all x and y

$|H_2| = (e^{x+y} + e^{x-y})^2 - (e^{x+y} - e^{x-y})^2$

$= \cancel{e^{2(x+y)}} + 2e^{2x} + \cancel{e^{2(x-y)}} - \cancel{e^{2(x+y)}} + 2e^{2x} - \cancel{e^{2(x-y)}}$

$= 4e^{2x} > 0$ for all x and y, Then the function of x and y is convex.

2.3) $F_x = e^{x+y} + e^{x-y} - \frac{3}{2}$; $e^{x+y} + e^{x-y} = \frac{3}{2}$ - (1)

$F_y = e^{x+y} - e^{x-y} - \frac{1}{2}$; $e^{x+y} - e^{x-y} = \frac{1}{2}$ - (2)

(1) + (2) ; $2e^{x+y} = 2$

$e^{x+y} = 1$

$\ln e^{x+y} = \ln(1)$

$(x+y)\ln(e) = 0$

$x+y = 0$; $y = -x$

Plug-in $x+y=0$ into (2) ; $e^0 - e^{x-y} = \frac{1}{2}$

$1 - e^{x-y} = \frac{1}{2}$

$0.5 = e^{x-y}$

$\ln(0.5) = \ln e^{x-y}$

$\ln(0.5) = x-y$

$\ln(0.5) = 2x$

$x = 0.5\ln(0.5)$

$y = -0.5\ln(0.5)$

$\therefore$ Critical point (x,y) is $(0.5\ln(0.5), -0.5\ln(0.5))$

Critical point is the local minimizer because the function is convex

Global minimum :

$e^{0.5\ln(0.5)-0.5\ln(0.5)} + e^{0.5\ln(0.5)+0.5\ln(0.5)} - \frac{3}{2}[0.5\ln(0.5)] - \frac{1}{2}[-0.5\ln(0.5)]$

$= e^0 + e^{\ln(0.5)} - \frac{3}{4}\ln(0.5) + \frac{1}{4}\ln(0.5)$

$= 1 + e^{\ln(0.5)} - \frac{1}{2}\ln(0.5)$ ※

3.1) What type of the return to scale technology does the production function exhibit?

From now on, assume that c = ¼ . Consider the following problems.

We add 't' into L & K

$$(tK)^{\frac{1}{3}}(tL)^{\frac{2}{3}} = t^{\frac{1}{3}}t^{\frac{2}{3}}K^{\frac{1}{3}}L^{\frac{2}{3}}$$

$$= tK^{\frac{1}{3}}L^{\frac{2}{3}}$$

$$= tQ$$

Degree of homogeneity is 1

Therefore, the profit function is constant return to scale.

From now on, assume that c = ¼ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$TR = PQ$	$TC = rK + wL$
$= Q^{-c}Q$	
$= Q^{-\frac{1}{4}+1}$	
$= (K^{\frac{1}{3}}L^{\frac{2}{3}})^{\frac{3}{4}}$	

Profit Function $= TR - TC$

$$= (K^{\frac{1}{3}}L^{\frac{2}{3}})^{\frac{3}{4}} - (rK + wL)$$

$$= (K^{\frac{1}{3}}L^{\frac{2}{3}})^{\frac{3}{4}} - rK - wL$$

$$= K^{\frac{1}{4}}L^{\frac{1}{2}} - rK - wL \quad //$$

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

$$\pi(K,L) = K^{\frac{1}{4}}L^{\frac{1}{2}} - rk - wL$$

F.O.C $\quad \frac{\partial \pi}{\partial k} = \frac{1}{4}K^{-\frac{3}{4}}L^{\frac{1}{2}} - r = 0$

$$L^{\frac{1}{2}} = 4r\,k^{\frac{3}{4}}$$

$$K = \left(\frac{L^{\frac{1}{2}}}{4r}\right)^{\frac{4}{3}} \text{ — } ①$$

$$\frac{\partial \pi}{\partial L} = \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}} - w = 0$$

$$K^{\frac{1}{4}} = 2wL^{\frac{1}{2}}$$

$$L = \left(\frac{K^{\frac{1}{4}}}{2w}\right)^{2} \text{ — } ②$$

Substitute ② into ①

$$K = \left[\frac{\left(\left(\frac{K}{2w}\right)^{2}\right)^{\frac{1}{2}}}{4r}\right]^{\frac{4}{3}}$$

$$K = \left[\frac{\left(\frac{K^{\frac{1}{4}}}{2w}\right)}{4r}\right]^{\frac{4}{3}}$$

$$K = \frac{K^{\frac{1}{3}}}{(2w)^{\frac{4}{3}}} \cdot \frac{1}{(4r)^{\frac{4}{3}}}$$

$$(8wr)^{\frac{4}{3}} = \frac{K^{\frac{1}{3}}}{K}$$

$$(8wr)^{\frac{4}{3}} = K^{-\frac{2}{3}}$$

$$K^{\frac{2}{3}} = \frac{1}{(8wr)^{\frac{4}{3}}}$$

$$K^{*} = \left[\frac{1}{(8wr)^{\frac{4}{3}}}\right]^{\frac{3}{2}}$$

$$= \frac{1}{(8wr)^{2}} \;\#$$

$$L^{*} = \left[\frac{\left(\frac{1}{8wr}\right)^{\frac{1}{4}}}{2w}\right]^{2}$$

$$= \frac{1}{8wr} \cdot \frac{1}{2w^{2}}$$

$$= \frac{1}{32w^{3}r} \;\#$$

The demand for factor inputs

- Labor Demand $\quad p \cdot MP_K = VMP_K = \frac{1}{8wr^{2}}$
- Capital Demand $\quad p \cdot MP_L = VMP_L = \frac{1}{32w^{3}r}$

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

From 3.3) $L^* = \dfrac{1}{32 w^3 r}$

$$\frac{\partial L^*}{\partial w} = -\frac{3}{32 w^4 r}$$

when $w \uparrow$ by 1 unit, optimal amount of labour or $L^* \downarrow$ by $\left(\dfrac{3}{32 w^4 r}\right)$ unit.

$$\frac{\partial L^*}{\partial r} = -\frac{1}{32 w^3 r^2}$$

when $r \uparrow$ by 1 unit, optimal amount of labour or $L^* \downarrow$ by $\left(\dfrac{1}{32 w^3 r^2}\right)$ unit.

3.5) Confirm your answer with the second-order condition.

$\pi_K = \frac{1}{4}K^{-\frac{3}{4}}L^{\frac{1}{2}} - r;\ \pi_L = \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}} - w$

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} & \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} \\ \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} & -\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}} \end{bmatrix}$$

$$|H_1| = \left| -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} \right| = -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}}$$

$$|H_2| = \begin{vmatrix} -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} & \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} \\ \frac{1}{8}K^{-\frac{3}{4}}L^{\frac{1}{2}} & -\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}} \end{vmatrix}$$

$$= \left(-\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}}\right)\left(-\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}}\right) - \left(\frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}\right)\left(\frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}\right)$$

$$= \frac{3}{64}K^{-\frac{3}{2}}L^{-1} - \frac{1}{64}K^{-\frac{3}{2}}L^{-1}$$

$$= \frac{1}{32}K^{-\frac{3}{2}}L^{-1}$$

H is negative definite at K^*, L^*

$d^2x < 0$ at K^*, L^*

$\therefore K^*, L^*$ is a local minimizer

Every input (K^*, L^*) will always give $|H_1| < 0$ and $|H_2| > 0$

meaning that d^2x is always less than 0.

$\therefore \pi$ is globally concave

$\therefore$ local minimizer = global minimizer.