

# Multivariable Integration

## 1. Indefinite Integral (Undefined limit, the answer is an equation.) [MA216]

From some of the differentiation rules, the following integration formulas can be derived:

2.1	$\int k dx = kx + C$ , $k$ is a constant
2.2	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ , $n \neq -1$
2.3	$\int e^x dx = e^x + C$
2.4	$\int kf(x) dx = k \int f(x) dx$ , $k$ is a constant
2.5	$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

**NOTE:** we can always check the integration result by differentiating it!

## 2. Power Rule for Integration and Integration by Substitution. [MA216]

$\therefore \int [u(x)]^n du = \frac{u^{n+1}}{n+1} + C$	<b>Power Rule</b>
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If $u$ is differentiable and a function of $x$ , then $\int u^n du = \frac{u^{n+1}}{n+1} + C$ if $n \neq -1$ .
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Let  $u$  be a function of  $x$ ,

$\int e^u du = e^u + C$
$\int \frac{1}{u} du = \ln u  + C$

**Ex.1:** Show that  $\int x^2(3x^3 + 7)^3 dx = \frac{(3x^3 + 7)^4}{36} + C$  **[Hint:  $u = 3x^3 + 7$ ]**

**Ex.2:** Show that

(a)  $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$ .

(b)  $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln|ax+b| + C$ .

**[Hint:  $u = ax + b$ ]**

### Integration Hints:

- Try to arrange the integrand into a familiar integration forms (seen formulas). If you can't find the right form, try some more! There is no shortcut! **ONLY PRACTICE!**
- When integrating fraction forms, sometimes a preliminary long division is needed to get familiar integration forms.

**Ex.3:** Show that  $\int \left( \frac{x+3}{x+6} \right) dx = x - 3 \ln|x+6| + C$ .

**Ex.4:** Show that  $\int \left( \frac{x^3 + x^2 - x - 3}{x^2 - 3} \right) dx = \frac{x^2}{2} + x + \ln|x^2 - 3| + C$ .

**Ex.5:** Calculate  $I = \int x^3 e^{4x^4} dx$ .

$$[\text{ANS: } I = \frac{1}{16} e^{4x^4} + C]$$

### 3. Definite Integral (With numerical limit, the answer is real number.) [MA216]

If  $F(x)$  is any anti-derivative of the function  $f(x)$ , the 'definite' integral of  $f(x)$  between  $a$  and  $b$  is

$$\int_a^b f(x) dx = F(b) - F(a) = [F(x)]_a^b \quad a < b$$

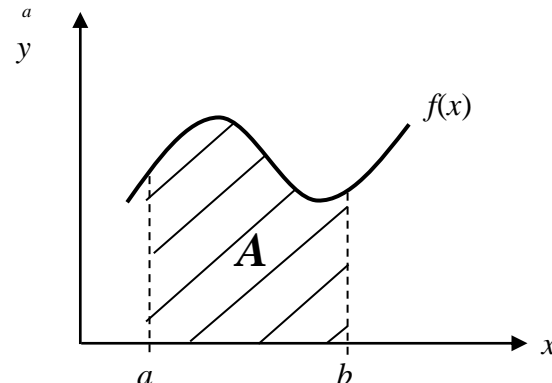
$a$  is the lower limit of integration and  $b$  is the upper limit of integration. **Definite** integral will give a **definite** numerical value, **NO x or C**.  $C$  will cancel itself out during the calculation.

**Ex.6:** Show that  $\int_1^5 3x^2 dx = 124$ .

When using integration by substitution for a definite integral, we need to **substitute the limit** as well or **substitute before taking limits**.

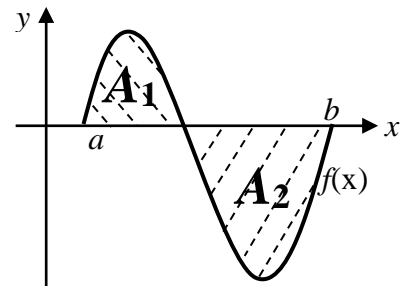
### 4. Definite Integral as an Area under a Curve [MA216]

$$A = \int_a^b f(x) dx = \text{Area under a curve between } x = [a, b]$$



**NOTE:** Definite integral **DOES NOT** always represents an area.

$\int_a^b f(x)dx = A_1 - A_2 = \text{NEGATIVE}$ . Area can't be negative!

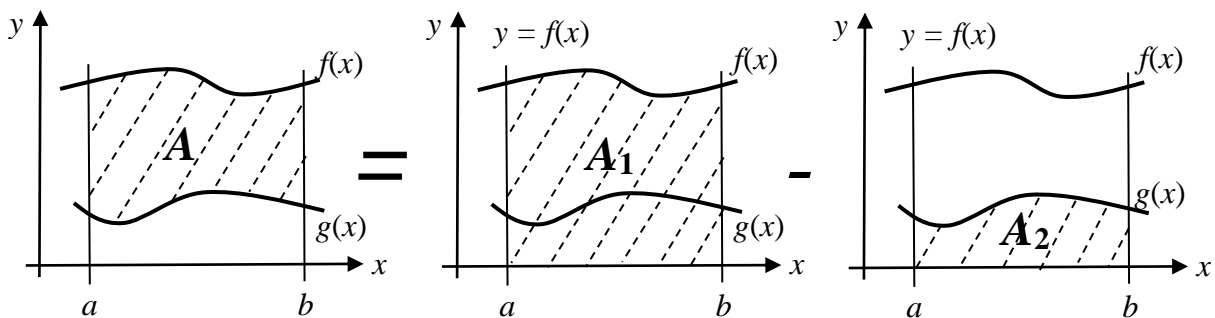


## 5. Fundamental Theorem of Integration Calculus

Hence, if  $f$  is continuous on interval  $[a, b]$ ,  $F(x)$  is any anti-derivative of  $f(x)$  on the interval then

$$\int_a^b f(x)dx = F(b) - F(a)$$

### 5.1 Area between Curves



Find area bounded by  $f(x)$ ,  $g(x)$ ,  $x = a$ , and  $x = b$ .

$$A = A_1 - A_2$$

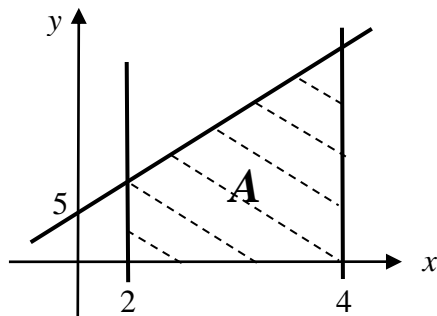
$$\therefore \int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

Similarly others properties of definite integral can be derived from area between curves.

**Ex.7:** Use a definite integral to find the area of the region bounded by the given curve, the  $x$  axis, and the given lines.

(Note: We need to sketch the curve to make sure that there will be no negative area!)

(i)  $y = x + 5, x = 2, \text{ and } x = 4$



$$A = \int_2^4 (x + 5) dx = \left[ \frac{x^2}{2} + 5x \right]_2^4$$

$$A = \left( \frac{4^2}{2} + 5(4) \right) - \left( \frac{2^2}{2} + 5(2) \right)$$

$$A = 28 - 12 = 16$$

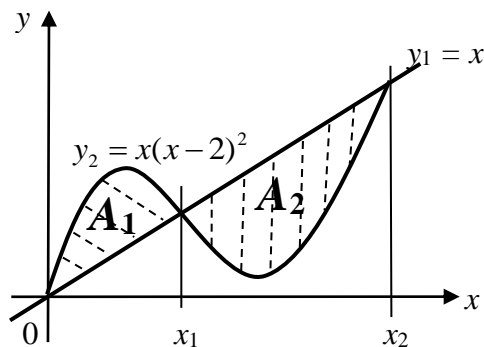
(ii)  $y = e^x, x = 1, \text{ and } x = 3$

[Ans:  $e^3 - e$ ]

(vi)  $y = \frac{1}{x}, x = 1, \text{ and } x = e^2$

[ANS: 2]

**Ex.8:** Express the area in terms of integrals



Find intersection points

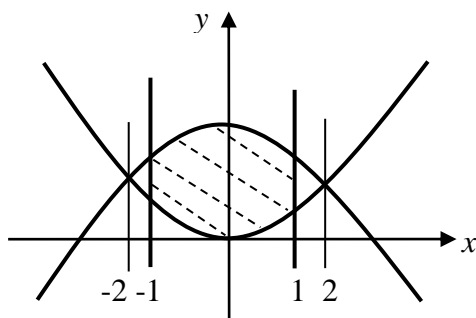
$$x(x-2)^2 = x$$

$A =$

**Ex.9:** Find the area of the region bounded by the graphs of the given equations. Be sure to find any needed points of intersection.

(i)  $y = 8 - x^2, y = x^2, x = -1, \text{ and } x = 1$

[Ans: 44/3]



Intersection:  $8 - x^2 = x^2$   
 $2x^2 = 8 \Rightarrow x = \pm 2$

$$A = \int_{-1}^1 [(8 - x^2) - x^2] dx = \frac{44}{3}$$

## 6. Integration Technique

In many cases, it is not possible to integrate a function by standard formulas. There are techniques that can be applied to change certain integrals to forms that are easier to integrate, for example, *integration by parts* and *integration by partial fractions*.

### 6.1 Integration by Parts

Integration by parts technique could become very useful if the integrand is the **product of two functions**.

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx} \quad (1) \text{ Given!}$$

Some books, written (1) in slightly different form.

$$\int u dv = uv - \int v du$$

**Hints:** Suppose  $H(x)$  is a product of two functions, to integrate  $H(x)$ , write it in the form of  $f(x)g'(x)$ . We need to choose  $f(x)$  and  $g'(x)$  very carefully so that we can calculate  $f'(x)$  and  $g(x)$ , and we can **easily** integrate  $\int f'(x)g(x)dx$ . Sometimes the **best choice** for  $f(x)$  and  $g'(x)$  is not obvious. Insight into making a good choice will come only with *practice* and, of course, *trial and error*.

Integration by parts also works for definite integral.

$$\boxed{\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx}$$

**Ex.10:** Find  $\int xe^x dx$ .

Solution:  $xe^x$  is a product of two functions so it is possible to use integration by parts.

Let's  $f(x) = x$        $g'(x) = e^x$

$f'(x) = 1$        $g(x) = \int e^x dx = e^x [ + C ]$       Will add constant of integration at the end.

**Recall:**  $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

$$\int xe^x dx = (x)(e^x) - \int (1)(e^x) dx$$

$$\int xe^x dx = xe^x - e^x + C$$

**Check:**  $\frac{d}{dx}(xe^x - e^x + C) = [(1)e^x + xe^x] - e^x + 0 = xe^x$

**Note:** Selecting  $f(x)$  and  $g'(x)$  badly will complicate the problem!

For example, let's  $f(x) = e^x$        $g'(x) = x$

$$f'(x) = e^x \quad g(x) = \frac{x^2}{2}$$

**Recall:**  $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

$$\int e^x x dx = e^x \left( \frac{x^2}{2} \right) - \int e^x \left( \frac{x^2}{2} \right) dx$$

$\int e^x \left( \frac{x^2}{2} \right) dx$  is in fact more complicated than the original integral  $\int xe^x dx$ .

**Carefully choose  $f(x)$  and  $g'(x)$ . Practice to gain more experience!**

**Ex.11:** Find  $I = \int \frac{1}{x} \cdot \ln x dx$ . [Ans:  $I = \frac{(\ln x)^2}{2} + C$ ]

### 6.2 Integration by Partial Fractions

Integration by partial fractions could become very useful if the integrand is a rational function in the form of a **quotient of two polynomials**  $N(x)$  and  $D(x)$  where they are polynomials.

The power of  $N(x)$  should be less than the power of  $D(x)$ . If **not**, we need to use long division first.

$$f(x) = \frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

**Ex.11:** Use long division to show that  $\frac{2x^4 - 3x^3 - 4x^2 - 17x - 6}{x^3 - 2x^2 - 3x} = 2x + 1 + \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x}$ .

**Ex.12:** Use integration by partial fractions to determine  $I = \int \left( \frac{2x^4 - 3x^3 - 4x^2 - 17x - 6}{x^3 - 2x^2 - 3x} \right) dx$

$$I = \int \left( \frac{2x^4 - 3x^3 - 4x^2 - 17x - 6}{x^3 - 2x^2 - 3x} \right) dx$$

$$I = \int \left( 2x + 1 + \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x} \right) dx$$

$$I = x^2 + x + \int \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x} dx$$

To integrate  $F$ , we need to factorise the denominator.

$$F = \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x} = \frac{4x^2 - 14x - 6}{x(x+1)(x-3)}$$

$x, (x+1), (x-3)$  are distinct linear factors and appear only **once**. Thus

$$F = \frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x-3)}$$

$$F = \frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{A(x+1)(x-3) + Bx(x-3) + Cx(x+1)}{x(x+1)(x-3)}$$

$$4x^2 - 14x - 6 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1) \quad [\text{eq. (1)}]$$

Find  $A, B$  and  $C$  from the above equation.

- 1) If  $x = 0$ ;  $0+0-6=A(1)(-3)+0+0 \rightarrow A = 2$   
 2) If  $x = -1$ ;  $4(-1)^2 - 14(-1) - 6 = 0 + B(-1)(-1-3) + 0 \rightarrow B = 3$   
 3) If  $x = 3$ ;  $4(3)^2 - 14(3) - 6 = 0 + 0 + C(x)(3+1) \rightarrow C = -1$

Hence,

$$\frac{4x^2 - 14x - 6}{x(x+1)(x-3)} = \frac{2}{x} + \frac{3}{x+1} - \frac{1}{x-3}$$

$$\int \frac{4x^2 - 14x - 6}{x^3 - 2x^2 - 3x} dx = \int \left( \frac{2}{x} + \frac{3}{x+1} - \frac{1}{x-3} \right) dx$$

$$= \int \left( \frac{2}{x} \right) dx + \int \left( \frac{3}{x+1} \right) dx - \int \left( \frac{1}{x-3} \right) dx$$

$$= 2 \ln|x| + 3 \ln|x+1| - \ln|x-3| + C$$

$$= \ln|x|^2 + \ln|x+1|^3 - \ln|x-3| + C$$

$$= \ln \left| \frac{x^2(x+1)^3}{x-3} \right| + C$$

**Note:** From eq.(1), there is an alternative method to find  $A$ ,  $B$  and  $C$  by comparing the coefficients on the left-hand-side and right-hand-side of the equation for each term.

**Recall eq.(1):**

$$4x^2 - 14x - 6 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1)$$

$$= A(x^2 - 2x - 3) + B(x^2 - 3x) + C(x^2 + x)$$

$$= Ax^2 - 2Ax - 3A + Bx^2 - 3Bx + Cx^2 + Cx$$

$$= (A + B + C)x^2 + (-2A - 3B + C)x + (-3A)$$

Coefficients of $x^2$ :	$4 = A + B + C$	}	$A = 2$
Coefficients of $x$ :	$-14 = -2A - 3B + C$		$B = 3$
Coefficients of constants:	$-6 = -3A$		$C = -1$

**Ex.13:** Calculate  $I = \int \frac{5x-2}{x^2-x} dx$ . [Ans:  $I = 2 \ln|x| + 3 \ln|x-1| + C$ ]

**Ex.14:** Calculate  $I = \int \frac{x+5}{x^2-1} dx$ . [Ans:  $I = 3 \ln|x-1| - 2 \ln|x+1| + C$ ]

### Repeated Linear Factors

For  $n$ th repeated linear factors, there will be the sum of  $k$  partial fractions.

$$\frac{N(x)}{(x-a)^k} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \dots + \frac{K}{(x-a)^k}$$

### Distinct Irreducible Quadratic Factors

$$\frac{N(x)}{(x^2 + bx + c)} = \frac{Ax + B}{(x^2 + bx + c)}$$

### Distinct Repeated Irreducible Quadratic Factors

$$\frac{N(x)}{(x^2 + bx + c)^k} = \frac{Ax + B}{(x^2 + bx + c)} + \frac{Cx + D}{(x^2 + bx + c)^2} + \dots + \frac{Yx + Z}{(x^2 + bx + c)^k}$$

**Ex.15:** Express  $y = \frac{3x^2 + 5}{(x^2 + 4)^2}$  in partial fractions. [Ans:  $\frac{3x^2 + 5}{(x^2 + 4)^2} = \frac{3}{(x^2 + 4)} - \frac{7}{(x^2 + 4)^2}$ ]

**Ex.16:** Determine  $I = \int \frac{-2x - 4}{x^3 + x^2 + x} dx$ . [Ans:  $I = -4 \ln|x| + 2 \ln|x^2 + x + 1| + C$ ]

**Ex.17:** Determine  $I = \int \frac{2x^2 - 5x - 2}{(x - 2)^2(x - 1)} dx$ . [Ans:  $I = \left(\frac{4}{x - 2}\right) + \ln\left|\frac{(x - 2)^7}{(x - 1)^5}\right| + C$ ]

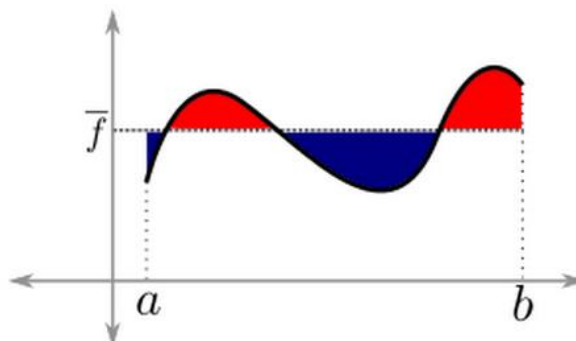
**Ex.18:** Determine  $I = \int \frac{2(x^2 + 8)}{x^3 + 4x} dx$ .

**Ex.19:** Determine  $I = \int \frac{2x^4 + 9x^2 + 8}{x(x^2 + 2)^2} dx$ . [Ans:  $I = \ln x^2 - \frac{1}{2(x^2 + 2)} + C$ ]

### 7. Average Value of a Function

The average (or mean) value of a function  $y = f(x)$  over the interval  $[a, b]$  is

$$\bar{f} = \frac{1}{b - a} \int_a^b f(x) dx$$



**Ex.20:** Find the average value of the function  $f(x) = x^2$  over the interval  $[1, 2]$ . [Ans: 7/3]

**Ex.21:** The profit of a business is given by

$$P = P(q) = 369q - 2.1q^2 - 400$$

where  $q$  is the number of units of the product sold. Find the average profit on the interval from  $q = 0$  to  $q = 100$ .

[Ans: 11,050]

**Ex.22:** A saving of 3,000 THB earns interest at an annual rate of 5% compounded continuously. After  $t$  years, its values  $S$  is given by  $S = 3000e^{0.05t}$ . Find the average value of a two-year saving.

[Ans: 3,155]

## 8. Multiple Integrals

The idea is to extend the idea of a definite integral (section 7) to double and triple integrals of functions of two or three variables.

### 8.1 Partial Antidifferentiation

It is possible to antidifferentiate a function of two or more variables with respect to one of the variables by treating all the other variables as though they are constants.

$$\int f(x, y) dx \Rightarrow \text{antidifferentiate } f(x, y) \text{ w.r.t. } x, \text{ holding } y \text{ constant.}$$

$$\int f(x, y) dy \Rightarrow \text{antidifferentiate } f(x, y) \text{ w.r.t. } y, \text{ holding } x \text{ constant.}$$

Note: The constant of integration is now a function of the 'constant' variable(s).

**Ex.23:** Find the following integrals

$$(a) \int (6xy^2 + 3x^2) dy \quad \text{and (b) } \int (6xy^2 + 3x^2) dx$$

$$[\text{Ans: (a) } 2xy^3 + 3x^2y + C(x) \text{ and (b) } 3x^2y^2 + x^3 + E(y)]$$

**Ex.24:** Find the following integrals

$$(a) \int (4xy + 12x^2y^2) dy \quad \text{and (b) } \int (4xy + 12x^2y^2) dx$$

$$[\text{Ans: (a) } 2xy^2 + 4x^2y^3 + C(x) \text{ and (b) } 2x^2y + 4x^3y^2 + E(y)]$$

**Ex.25:** Find the following integrals

$$(a) \int_0^2 (6xy^2 + 3x^2) dy \quad \text{and (b) } \int_0^1 (6xy^2 + 3x^2) dx$$

$$[\text{Ans: (a) } 16x + 6x^2 \text{ and (b) } 3y^2 + 1]$$

**Ex.26:** Find the following integrals

$$(a) \int_0^1 (4xy + 12x^2y^2) dy \quad \text{and (b) } \int_0^3 (4xy + 12x^2y^2) dx$$

$$[\text{Ans: (a) } 2x + 4x^2 \text{ and (b) } 18y + 108y^2]$$

**Ex.27:** Find the following integrals

$$(a) \int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and (b) } \int_1^2 \int_0^3 x^2 y \, dx \, dy$$

**Ex. 27(a):** Regarding  $x$  as a constant, we obtain

$$\begin{aligned}\int_1^2 x^2 y \, dy &= \left[ x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} \\ &= x^2 \left( \frac{2^2}{2} \right) - x^2 \left( \frac{1^2}{2} \right) \\ &= \frac{3}{2} x^2\end{aligned}$$

We now integrate the function of  $x$  from 0 to 3 considering  $y$  as a constant.

$$\begin{aligned}\int_0^3 \int_1^2 x^2 y \, dy \, dx &= \int_0^3 \left( \int_1^2 x^2 y \, dy \right) dx = \int_0^3 \left( \frac{3}{2} x^2 \right) dx \\ &= \left[ \frac{x^3}{2} \right]_{x=0}^{x=3} \\ &= \frac{27}{2}\end{aligned}$$

**Ex. 27 (b):** We first integrate the function of  $x$  from 0 to 3 considering  $y$  as a constant.

$$\begin{aligned}\int_1^2 \int_0^3 x^2 y \, dx \, dy &= \int_1^2 \left( \int_0^3 x^2 y \, dx \right) dy \\ &= \int_1^2 \left[ \frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\ &= \int_1^2 9y \, dy \\ &= \left[ 9 \frac{y^2}{2} \right]_{y=1}^{y=2} \\ &= \frac{27}{2}\end{aligned}$$

**Ex.28:** Find the following integrals

$$(a) \int_0^1 \int_0^2 (6xy^2 + 3x^2) \, dy \, dx \quad \text{and (b) } \int_0^2 \int_0^1 (6xy^2 + 3x^2) \, dx \, dy$$

[Ans: 10]

**Ex.29:** Find the following integrals

$$(a) \int_0^3 \int_0^1 (4xy + 12x^2 y^2) \, dy \, dx \quad \text{and (b) } \int_0^1 \int_0^3 (4xy + 12x^2 y^2) \, dx \, dy$$

[Ans: 45]

## 8.2 Volume (Double Integral)

In a similar manner, we consider a function  $f$  of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d]$$

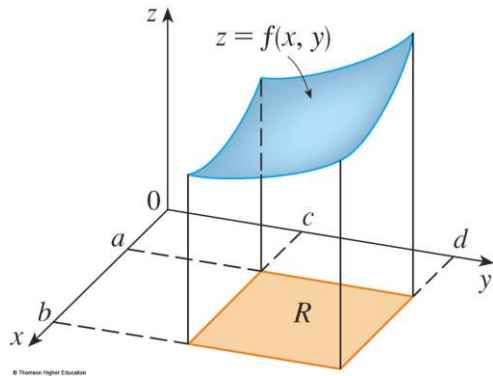
$$= \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

and we first suppose that  $f(x, y) \geq 0$ .

The graph of  $f$  is a surface with equation  $z = f(x, y)$ .

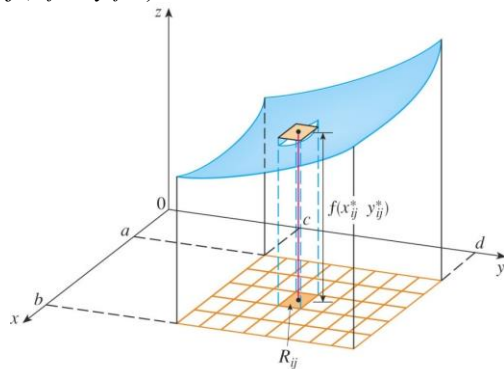
Let  $S$  be the solid that lies above  $R$  and under the graph of  $f$ , that is,

$$S = \{(x, y, z) \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$



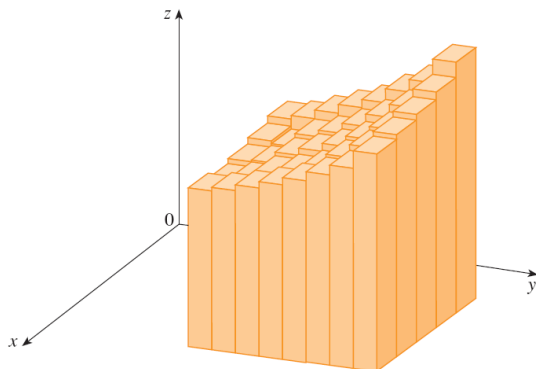
The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*) \Delta A$$



Thus, we get an approximation to the total volume of  $S$ :

$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$



Our intuition tells us that the approximation  $V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$  becomes better as

$m$  and  $n$  become larger.

So, we would expect that:

$$V = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A$$

The double integral of  $f$  over the rectangle  $R$  is:

$$\iint_R f(x, y) dA = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A \quad \text{if this limit exists.}$$

If  $f(x, y) \geq 0$ , then the volume  $V$  of the solid that lies above the rectangle  $R$  and below the surface  $z = f(x, y)$  is:

$$\begin{aligned} \iint_R f(x, y) dA &= \int_a^b \left[ \int_c^d f(x, y) dy \right] dx \\ &= \int_c^d \left[ \int_a^b f(x, y) dx \right] dy \end{aligned}$$

**Ex.30:** Find the double integral

$$\iint_R (x - 3y^2) dA \quad \text{over } R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$$

$$\begin{aligned} \iint_R (x - 3y^2) dA &= \int_0^2 \int_1^2 (x - 3y^2) dy dx \\ &= \int_0^2 [xy - y^3]_{y=1}^{y=2} dx \\ &= \int_0^2 (x - 7) dx \\ &= \left[ \frac{x^2}{2} - 7x \right]_0^2 \\ &= -12 \end{aligned}$$

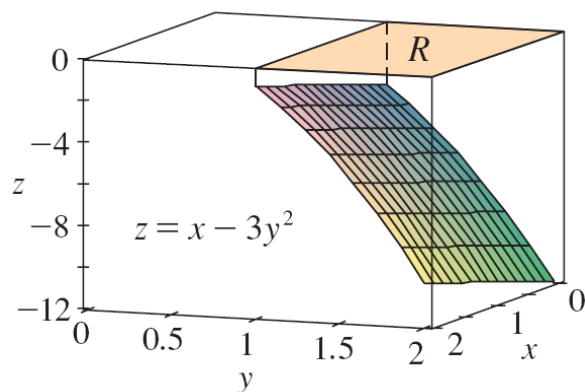
$$\text{Try } \iint_R (x - 3y^2) dA = \int_1^2 \int_0^2 (x - 3y^2) dx dy = -12.$$

Notice the negative answer in this example.

Nothing is wrong with that.

- The function  $f$  in the example is not a positive function.
- So, its integral doesn't represent a volume.

**Ex.31:** Evaluate  $\iint_R (x + y) dA$  over  $R = \{(x, y) \mid 1 \leq x \leq 3, -1 \leq y \leq 2\}$



[Ans: 15]

**Ex.32:** Find the volume of the solid under the graph  $f(x, y) = 1 + x^2 + y^2$  over the rectangular  $R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$

[Ans: 5/3 cubic units]

**Ex.33:** Find the volume of the solid under the graph  $f(x, y) = x + 2y$  over the rectangular  $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$

[Ans: 4 cubic units]

**Ex.34:** Evaluate  $\iint_R 2xe^{x^2+y} dA$  over  $R = \{(x, y) \mid 0 \leq x \leq 1, -1 \leq y \leq 1\}$

[Ans:  $e^2 - e - 1 + e^{-1}$ ]

**Ex.35:** Evaluate  $\iint_R \frac{x}{y^2} e^{\frac{x}{y}} dA$  over  $R = \{(x, y) \mid 0 \leq x \leq 1, 1 \leq y \leq 2\}$

[Ans:  $e - 2e^{\frac{1}{2}} + 1$ ]

**Ex.36:** Evaluate  $\int_0^{\infty} \left[ \int_0^{\infty} y(0.4e^{-x}e^{-0.4y}) dx \right] dy$

[Ans: 2.5]

### 8.3 Average Value of a Function (Extension)

Similarly to section 11, we define the average value of a function  $f$  of two variables defined on a rectangle  $R$  to be:

$$\bar{f} = \frac{1}{A(R)} \iint_R f(x, y) dA \text{ if } f(x, y) \geq 0$$

where  $A(R)$  is the area of  $R$ .

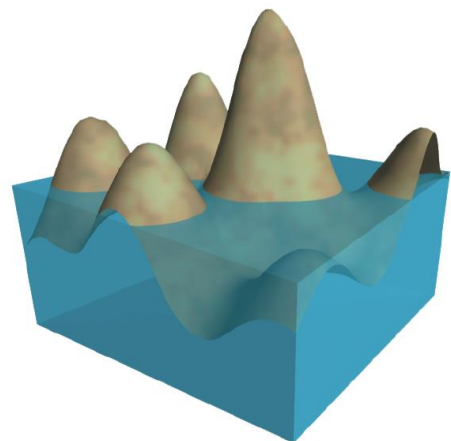
$$\text{Hence, } \bar{f} = \frac{1}{(b-a)(d-c)} \iint_R f(x, y) dA$$

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\} \text{ if } f(x, y) \geq 0$$

$$\bar{f} \times A(R) = \iint_R f(x, y) dA$$

The box with base  $R$  and height  $\bar{f}$  has the same volume as the solid that lies under the graph of  $f$ .

Suppose  $z = f(x, y)$  describes a mountainous region and you chop off the tops of the mountains at height  $\bar{f}$ . Then, you can use them to fill in the valleys so that the region becomes completely flat.



**Ex.37:** Find the average value of  $f(x, y) = 4 - \frac{1}{2}x - \frac{1}{2}y$  over the rectangular  
 $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$

[Ans: 3]

**Ex.38:** Find the average value of  $f(x, y) = x + 2y$  over the rectangular  
 $R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$

[Ans: 2]

### 8.4 Properties of Double Integrals

These properties are referred to as the linearity of the integral.

$$\iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA$$

$$\iint_R cf(x, y) dA = c \iint_R f(x, y) dA$$

where  $c$  is a constant

If  $f(x, y) \geq g(x, y)$  for all  $(x, y)$  in  $R$ , then

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$$

**Ex.39:** In a certain factory, output is given by the Cobb-Douglas production function

$$Q(K, L) = 50K^{\frac{3}{5}}L^{\frac{2}{5}}$$

where  $K$  is the capital investment in units of 1,000,000 THB and  $L$  is the size of the labour force measured in worker-hours. Suppose that monthly capital investment varies between 10,000,000 THB and 12,000,000 THB, while monthly use of labour varied between 2,800 and 3,200 worker-hours. Find the average monthly output for the factory.

[Ans: 5,181 unit]

### 8.5 Double Integrals with non-rectangular limit

It is also possible to have non-rectangular limit.

**Ex.40:** Determine  $I = \int_0^1 \int_{x^3}^{x^2} (x^3 - xy) dy dx$

$$\begin{aligned} \int_{x^3}^{x^2} (x^3 - xy) dy &= \left[ x^3 y - \frac{x}{2} y^2 \right]_{y=x^3}^{y=x^2} \\ &= \left[ x^3 x^2 - \frac{x}{2} (x^2)^2 \right] - \left[ x^3 x^3 - \frac{x}{2} (x^3)^2 \right] \\ &= \left[ x^5 - \frac{x^5}{2} \right] - \left[ x^6 - \frac{x^7}{2} \right] \\ &= \frac{x^5}{2} - x^6 + \frac{x^7}{2} \end{aligned}$$

$$\begin{aligned} I &= \int_0^1 \left( \frac{x^5}{2} - x^6 + \frac{x^7}{2} \right) dx \\ &= \left[ \frac{x^6}{12} - \frac{x^7}{7} + \frac{x^8}{16} \right]_0^1 \\ &= \left[ \frac{1^6}{12} - \frac{1^7}{7} + \frac{1^8}{16} \right] - \left[ \frac{0^6}{12} - \frac{0^7}{7} + \frac{0^8}{16} \right] = \frac{1}{336} \end{aligned}$$

**Ex.41:** Determine  $I = \int_{-1}^1 \int_0^{1-x} (2x+1) dy dx$

[Ans: 2/3]

**Ex.42:** Determine  $I = \int_1^{\ln 22} \int_{e^y} dx dy$

[Ans:  $\ln 4 - 4 + e$ ]

## 8.5 Triple Integrals

It is also possible to have the integrals of 3 variables or more.

**Ex.43:** Determine  $I = \int_{-1}^0 \int_{-1}^2 \int_1^2 6xy^2z^3 dx dy dz$

$$\begin{aligned} \int_1^2 6xy^2z^3 dx &= \left[ 3x^2y^2z^3 \right]_{x=1}^{x=2} \\ &= 3y^2z^3 \left[ x^2 \right]_{x=1}^{x=2} \\ &= 3y^2z^3 [2^2 - 1^2] \\ &= 3y^2z^3 [4 - 1] \\ &= 9y^2z^3 \end{aligned}$$

$$\begin{aligned} \int_{-1}^2 \int_1^2 6xy^2z^3 dx dy &= \int_{-1}^2 9y^2z^3 dy \\ &= \left[ 3y^3z^3 \right]_{y=-1}^{y=2} \\ &= 3z^3 \left[ y^3 \right]_{y=-1}^{y=2} \\ &= 3z^3 [2^3 - (-1)^3] \\ &= 3z^3 [8 + 1] \\ &= 27z^3 \end{aligned}$$

$$\begin{aligned} I &= \int_{-1}^0 27z^3 dz \\ &= \left[ \frac{27}{4} z^4 \right]_{z=-1}^{z=0} \\ &= \frac{27}{4} \left[ z^4 \right]_{z=-1}^{z=0} \\ &= \frac{27}{4} [0^4 - (-1)^4] \\ &= -\frac{27}{4} \end{aligned}$$

**Ex.44:** Determine  $I = \int_0^1 \int_0^x \int_0^{x-y} x dz dy dx$

[Ans: 1/8]

## Tutorial Sheet: Integration

1. Determine  $I = \int_{-1}^5 \int_1^3 (y+1) dx dy$  [Ans: 36]
2. Determine  $I = \int_0^1 \int_0^2 xy^2 dx dy$  [Ans: 2/3]
3. Determine  $I = \int_1^2 \int_0^2 e^{-(x+y)} dx dy$  [Ans:  $e^{-4} - e^{-3} - e^{-2} + e^{-1}$ ]
4. Determine  $I = \int_0^4 \int_{y/2}^2 xy dx dy$  [Ans: 8]
5. Determine  $I = \int_0^1 \int_{\sqrt{x}}^{x^2} 7(x^2 + 2xy - 3y^2) dy dx$  [Ans: 1/30]
6. Determine  $I = \int_0^1 \int_1^2 \frac{x}{y^2} e^{\frac{x}{y}} dy dx$  [Ans:  $e - 2e^{\frac{1}{2}} + 1$ ]
7. Determine  $I = \int_0^1 \int_{x^2}^x \int_0^{xy} dz dy dx$  [Ans: 1/24]
8. Determine  $I = \int_0^1 \int_y^{y^2} \int_0^1 9x^2 yz^2 dz dx dy$  [Ans: -3/40]
9. Determine  $I = \int_0^1 \int_0^1 \int_0^{x^2} (x^2 + y^2) dz dy dx$  [Ans: 14/45]