MA 332 LINEAR ALGEBRA

Course Details

Semester: 1/2016

Class hour: Monday 8.00-11.00 pm

Lecturer: Assoc. Prof. Julaluk Carmai, DPhil.

Email: julaluk.c.ae@tggs.kmutnb.ac.th or jc211cb@gmail.com

Midterm: Monday 3 October 2016 8.00-10.00 pm

Final: Friday 9th December 2015 9.00-12.00 am

MA 332 LINEAR ALGEBRA

Reference books:

- Linear algebra and its applications by Gilbert Strang ,
  International Thomson Publishing; 3rd edition (1988) ISBN: 0155510053.
- Linear algebra and its applications by David C. Lay , 3rd edition ,
  Addison Wesley
- Linear algebra with applications 4th edition by W. Keith Nicholson,
  McGraw Hill, 2002
- Elementary Linear algebra 7th edition by Bernard Kolman and David R. Hill,
  Prentice Hall.

Course grading:

Quizzes	10%
Mid-term examination	40 %
Final examination	50%

MA 332 LINEAR ALGEBRA

Course Outline

- Introduction to system of linear equations
- Matrix Operations
- Solving system of linear equations
- Vector Spaces and Subspaces
- Orthogonality
- Determinants
- Eigenvalues and Eigenvectors
- Quadratic Functions and Quadratic Forms
- Linear Programming

MA 332 LINEAR ALGEBRA

Lecture I

- Linear equations and linear systems
- Systems of linear equations
- The geometry of linear equations
- Matrix Operations
- Elementary row operations

Linear equations and Linear Systems

A **linear equation:**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

EXAMPLE:

$4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$

↓ ↓

rearranged rearranged

↓ ↓

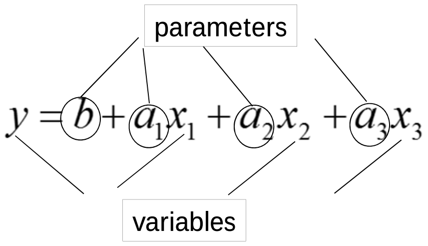
$3x_1 - 5x_2 = -2$ $2x_1 + x_2 - x_3 = 2\sqrt{6}$

Not linear:

$4x_1 - 6x_2 = x_1x_2$ and $x_2 = 2\sqrt{x_1} - 7$

Linear equations and Linear Systems

Economics models typically consist of a number of equations that represent identities, behavioral relationships and conditions that contribute an equilibrium. These equations include both variables, which are economics quantities that can assume different values and parameters, which are unvarying constants for example



parameters

$$y = \bigcirc\!\!\!\!b + \bigcirc\!\!\!\!\!a_1 x_1 + \bigcirc\!\!\!\!\!a_2 x_2 + \bigcirc\!\!\!\!\!a_3 x_3$$

variables

Linear equations and Linear Systems

A collection of one or more linear equations involving the same set of variables.

A system of linear equations with 2 variables:

$$\begin{aligned} ax+by&=h \\ cx+dy&=k \end{aligned} \qquad \text{E.g.} \quad \begin{aligned} 2x+y&=8 \\ x+3y&=9 \end{aligned}$$

A system of linear equations with 3 variables:

$$\begin{aligned} 6x_1+x_2+x_3&=6 \\ 5x_1+x_2+2x_3&=4 \\ 4x_1+x_2-x_3&=-2 \end{aligned}$$

Linear equations and Linear Systems

A system of linear equations with n variables:

$$\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots.a_{1n}x_n&=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots.a_{2n}x_n&=b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3+\ldots\ldots.a_{3n}x_n&=b_3 \\ . \qquad\qquad\qquad\qquad\qquad\qquad &\quad . \\ . \qquad\qquad\qquad\qquad\qquad\qquad &\quad . \\ . \qquad\qquad\qquad\qquad\qquad\qquad &\quad . \\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\ldots\ldots.a_{nn}x_n&=b_n \end{aligned}$$

Set variables: $x_1, x_2, x_3, \ldots\ldots x_n$

Linear equations and Linear Systems

Example of linear equations

A charity wishes to endow a fund that will provide $50000 per year for cancer research. The charity has $480000 and , to reduce risk, wants to invest in two banks paying 10% and 11 % respectively.
How much should be invested in each bank?

$$\begin{aligned} x + \quad\;\; y &= 480000 \\ 0.1\,x + 0.11\,y &= 50000 \end{aligned}$$

$$\begin{aligned} 10x+10y &= 4800000 \\ 10x+11y &= 5000000 \end{aligned}$$

$$\begin{aligned} y &= 200000 \\ x &= 280000 \end{aligned}$$

Applications to Economics

The application of linear algebra to economics lies primarily in its use of matrices.

A matrix in economics is used as a means to solve a large number of linear equations at once, where the variables are economic indicators and factors.

As a whole, then, a matrix represents a transformation from one state to another state, and one can view the economy as a succession of such states.

Given the huge number of factors involved, linear algebra has various methods
for reducing the complexity of the problem.

It also investigates properties of matrices such that one need not always waste time trying to find the precise solutions in order to determine some property of the system.

The input-output Model

One of the widely used model to analyze the economy is
The Leontief "input-output" model (or Production model)

- divided U.S. economy into 500 sectors (e.g. coal industry, automotive industry, communications)
- for each sector, wrote linear equation describing how sector distributes output to other sectors

Simpler form the Leontief input-output model is an exchange model.

Linear equations and Linear Systems

A Homogeneous system in Economics

An exchange model (a simpler form of the Leontief input-output mode)l

- Suppose a nation's Economy is divided into many sectors
  (e.g. various manufacturing, communication
  Entertainment agriculture and service industries)

Suppose

- For each sector we know its output per year and
- know how this output is divided or exchange among the other sectors in the economy

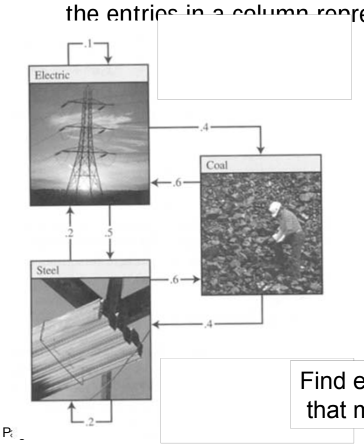
- **The total dollar value of a sector output= price of that output**

There exist equilibrium prices that can be assigned to the total outputs of various sectors in such a way that **the income of each sector exactly balances its expenses.**

Linear equations and Linear Systems

Example of how to find the equilibrium prices

Suppose an economy consists of the Coal, Electric (power), and steel sectors And the output of each sector is distributed among the various sectors, where the entries in a column represent the factorial parts of a sector's total output



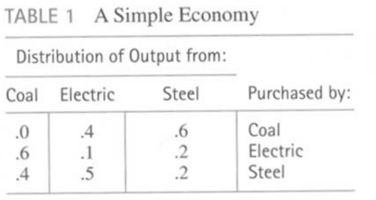


TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

Find equilibrium price (total output in dollar) that make each sector's income match its expenditure

---

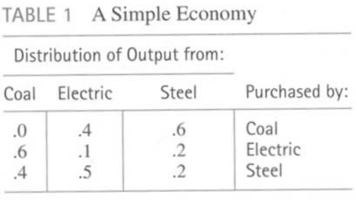


TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	Electric	Steel	Purchased by:
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

$$p_C = .4p_E + .6p_S$$

$$p_E = .6p_C + .1p_E + .2p_S$$

$$p_S = .4p_C + .5p_E + .2p_S$$

$$\begin{aligned} p_C - .4p_E - .6p_S &= 0 \\ -.6p_C + .9p_E - .2p_S &= 0 \\ -.4p_C - .5p_E + .8p_S &= 0 \end{aligned}$$

Markov Chain

EXAMPLE we examined a model for population movement between a city and its suburbs. See Fig. 1. The annual migration between these two parts of the metropolitan region was governed by the *migration matrix M*:

$$M = \begin{array}{cc|l} \multicolumn{2}{c}{\text{From:}} & \\ \text{City} & \text{Suburbs} & \text{To:} \\ \end{array}$$

$$M = \left[\begin{array}{cc} .95 & .03 \\ .05 & .97 \end{array}\right] \begin{array}{l} \text{City} \\ \text{Suburbs} \end{array}$$

That is, each year 5% of the city population moves to the suburbs, and 3% of the suburban population moves to the city. The columns of M are probability vectors, so M is a stochastic matrix. Suppose the 2000 population of the region is 600,000 in the city and 400,000 in the suburbs. Then the initial distribution of the population in the region is given by $\mathbf{x}_0$ in (1) above. What is the distribution of the population in 2001? In 2002?

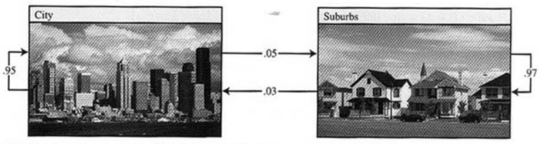


FIGURE 1 Annual percentage migration between city and suburbs.

Solution we saw that after one year, the population vector $\begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$ changed to

$$\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix} = \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix}$$

If we divide both sides of this equation by the total population of 1 million, and use the fact that $kM\mathbf{x} = M(k\mathbf{x})$, we find that

$$\begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .600 \\ .400 \end{bmatrix} = \begin{bmatrix} .582 \\ .418 \end{bmatrix}$$

The vector $\mathbf{x}_1 = \begin{bmatrix} .582 \\ .418 \end{bmatrix}$ gives the population distribution in 2001. That is, 58.2% of the region lived in the city and 41.8% lived in the suburbs. Similarly, the population

Markov Chain

EXAMPLE: Suppose that 3% of the population of the U.S. lives in the State of Washington. Suppose the migration of the population into and out of Washington State will be constant for many years according to the following migration probabilities. What percentage of the total U.S. population will eventually live in Washington?

From :		
WA	Rest of U.S.	**To**:
.9	.01	WA
.1	.99	Rest of U.S.

Linear Programming

Example

Suppose General motors makes a profit of \$100 on each Chevrolet, \$200 on each Buick, and \$400 on each Cadillac. They get 20,17 and 14 miles per gallon, respectively, and Congress insists that the average car produced must get 18. The plant can assemble a Chevrolet in 1 minute, a Buick in 2 minutes, and a Cadillac in 3 minutes. What is the maximum profit in an 8-hour day?

Problem:
Maximize $100x + 200y + 400z$

subject to

$$\begin{aligned} &20x + 17y + 14z \geq 18(x + y + z) \\ &x + 2y + 3z \leq 480 \\ &x, y, z \geq 0 \end{aligned}$$

Systems of linear equations

A collection of one or more linear equations involving the same set of variables, say $x_1, x_2, x_3, \ldots\ldots x_n$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots a_{3n}x_n &= b_3 \\
. \qquad\qquad\qquad\qquad &\quad . \\
. \qquad\qquad\qquad\qquad &\quad . \\
. \qquad\qquad\qquad\qquad &\quad . \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots a_{nn}x_n &= b_n
\end{aligned}$$

$$\begin{aligned}
-3x + 4y &= 8 \\
6x + 10y &= 5
\end{aligned}$$

$$\begin{aligned}
3x_1 + 4x_2 &= 1 \\
x_1 - 2x_2 &= 7
\end{aligned}$$

$$\begin{aligned}
3x + y + z &= 2 \\
x - y + z &= 4 \\
5x - y + 2z &= 12
\end{aligned}$$

A solution of a linear system:

A list $s_1, s_2, s_3, \ldots\ldots s_n$ of numbers that makes each equation in the system true when the values $s_1, s_2, s_3, \ldots\ldots s_n$ are substituted for $x_1, x_2, x_3, \ldots\ldots x_n$ respectively

The solution set:

The set of all possible solutions of a linear system.

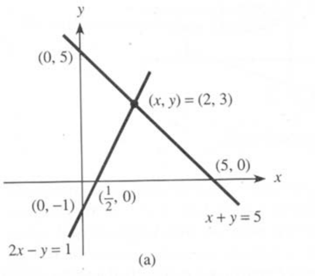
Row pictures
Column pictures
Matrix form

The geometry of linear equations

$$2x - y = 1$$
$$x + y = 5$$

(a) Look at an equation at a time

2 straight lines in xy plane



The row picture

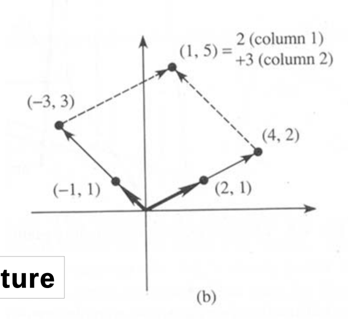
The geometry of linear equations

(b) Look at the columns of this linear system

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$$

A vector equation

A parallelogram



The combination of the column vectors on the left side

produces

The vector on the right

The column picture

The geometry of linear equations

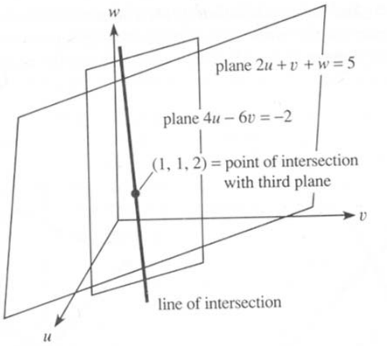
$$\begin{aligned} 2u+v+w&=5\\ 4u-6v&=-2\\ -2u+7v+2w&=9 \end{aligned}$$

The row picture:

Planes in 3D

If 2 equations in 3D intersect ➔______

If 3 equations in 3D intersect➔______



n equations , n unknowns
n-1 dimension planes in an n dimensional space

The column picture:

A vector equation, each vector with 3 components

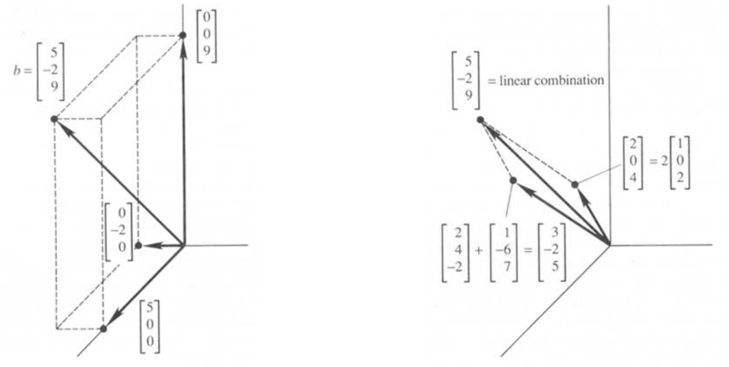
$$u\begin{bmatrix}2\\4\\-2\end{bmatrix}+v\begin{bmatrix}1\\-6\\7\end{bmatrix}+w\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}$$

3 vectors on the left combined to give a vector on the right.

↓

Multiplication by a scalar and addition of vectors

$$\begin{bmatrix}2\\4\\-2\end{bmatrix}+\begin{bmatrix}1\\-6\\7\end{bmatrix}+2\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}$$



n equations , n unknowns
A linear combination of the n vectors that equals *b*

Row picture: Intersection of n planes

Column picture: The right side *b* is a combination of the column vectors

Solution to equations:

Intersection point of planes= Coefficient in the combination of columns

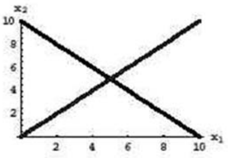
A solution of the system of algebraic equations

⇩

A list of numbers that makes each equation true

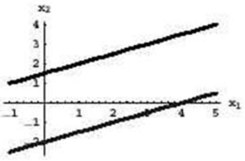
A system of linear equations has either

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$



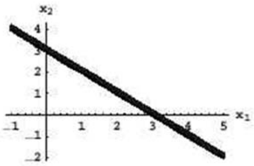
one unique solution

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$



no solution

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

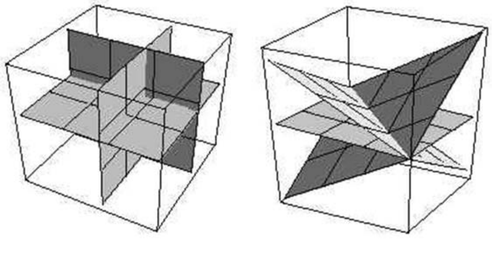


infinitely many solutions

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

i) The planes intersect in one point. *(one solution)*

ii) The planes intersect in one line. *(infinitely many solutions)*



Consistent system ➔ one solution or infinitely many solutions

Inconsistent system ➔ no solution

The singular cases

3 planes in 3D do not intersect! What can go wrong?

(a) Two planes may be parallel

two parallel

(b) Every pair of planes intersects
but no point is common to all three

no intersection

$$u+v+w=2$$
$$2u+3w=5$$
$$3u+v+4w=6$$

0=1 ➔ inconsistent

(c) An infinity of solutions is also another singular system

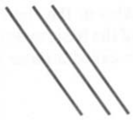
$$u+v+w=2$$
$$2u+3w=5$$
$$3u+v+4w=7$$

0=0

A whole line in common

line of intersection

(d) All 3 planes are parallel



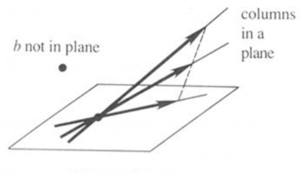
all parallel

For most right side ➔ no solution
For special right sides (b=(0,0,0))➔ The whole plane of solutions
(all 3 planes become the same)

What happens to the column picture when the system is singular?

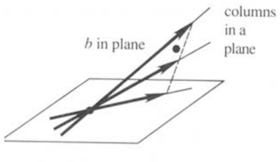
$$u\begin{bmatrix}1\\2\\3\end{bmatrix}+v\begin{bmatrix}1\\0\\1\end{bmatrix}+w\begin{bmatrix}1\\3\\4\end{bmatrix}=b$$

3 columns lie in a plane➔ all their linear combinations are also in the plane



(a) no solution

$$b=\begin{bmatrix}2\\5\\6\end{bmatrix}$$



(b) infinity of solutions

$$b=\begin{bmatrix}2\\5\\7\end{bmatrix}$$

If the *n* planes have no point in common,
then the *n* columns lie in the same plane

If the row picture breaks down so does the column picture

MATRIX

A powerful set of tools for handling systems of equations is provided by Matrix Algebra (Linear Algebra).

What is a matrix?

A Matrix is a rectangular (or square) array of numbers, variables or parameters with one or more rows and one or more columns

$$\begin{array}{c|cccc} & \text{Column 1} & \text{C 2} & \text{C 3} & \text{C 4} \\ \hline \text{Row 1} & 1 & -2 & 1 & 0 \\ \text{Row 2} & 0 & 2 & -8 & 8 \\ \text{Row3} & -4 & 5 & 9 & -9 \end{array}$$

Size of matrix is 3x4

entries/elements

$$\begin{bmatrix} 6 & 1 & 1 \\ 5 & 1 & 2 \\ 4 & 1 & -1 \end{bmatrix}$$

A 3x3 square matrix

A matrix with one row ➔ $\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & 12 \end{bmatrix}$

A row matrix or a row vector

A matrix with one column ➔ $\mathbf{A} = \begin{bmatrix} 1 \\ 34 \\ 9 \\ 17 \end{bmatrix}$

A column matrix or a column vector

Matrix notation

Two way to denote $m \times n$ matrix A:

In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Scalar entry in the ith row and jth column of $\boldsymbol{A}$ is denoted by a_{ij}

In terms of the *columns* of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

The columns of A are vectors in R^m

Main diagonal entries:__________________

Matrix notation

A diagonal matrix is a square matrix whose nondiagonal entries are zero

Identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Matrix addition and scalar multiplication

Two matrices are **equal** if they have the same size and if their corresponding entries are equal.

Matrix addition

- Matrices must have the same size
- The resulting matrix is also of the same size
- Each entry in ***A+B*** is the sum of the corresponding entries in ***A*** and ***B.***

Scalar multiplication

The scalar multiple *r**A*** is the matrix whose columns are *r* times the corresponding columns in ***A***

EXAMPLES

$$\mathbf{A}=\begin{bmatrix}4 & 0 & 5\\ -1 & 3 & 2\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}1 & 1 & 1\\ 3 & 5 & 7\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}2 & -3\\ 0 & 1\end{bmatrix}$$

$\mathbf{A}+\mathbf{B}=$

$\mathbf{A}+\mathbf{C}=$

$2\mathbf{B}=$

$\mathbf{A}-2\mathbf{B}$

Properties of Sums and Scalar Multiples

THEOREM 1

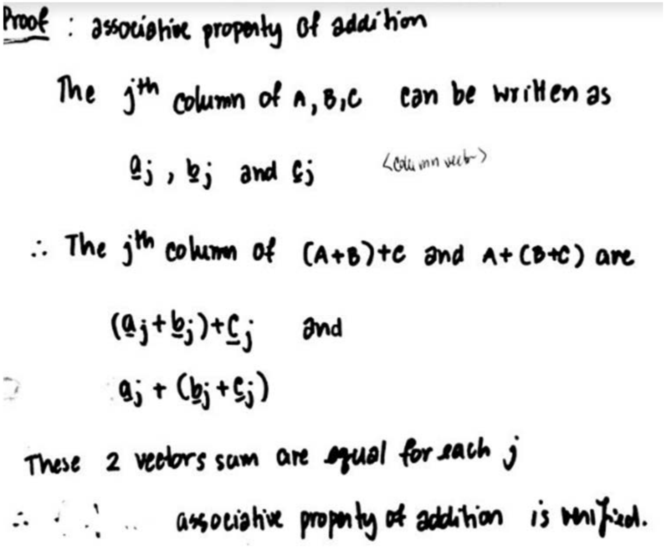
Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

Commutative law	a. $A + B = B + A$	d. $r(A + B) = rA + rB$	Distributive laws
Associative law of addition	b. $(A + B) + C = A + (B + C)$	e. $(r + s)A = rA + sA$	
	c. $A + 0 = A$	f. $r(sA) = (rs)A$	

Each equality is verified by showing that

- LHS matrix have the same size as RHS matrix
- corresponding columns are equal

Proof: Associative Property of addition



Proof : associative property of addition

The j^{th} column of A, B, C can be written as

$\underline{a}_j$, $\underline{b}_j$ and $\underline{c}_j$ <column vector>

∴ The j^{th} column of $(A+B)+C$ and $A+(B+C)$ are

$(\underline{a}_j+\underline{b}_j)+\underline{c}_j$ and

$\underline{a}_j + (\underline{b}_j+\underline{c}_j)$

These 2 vectors sum are equal for each j

∴ associative property of addition is verified.

Matrix multiplication

Three different ways with the same answer:

Method 1: Each entry of ***AB*** is the product of a row and a column.

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \\ \end{bmatrix}\begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} \\ & (AB)_{ij} & \\ \\ \end{bmatrix}$$

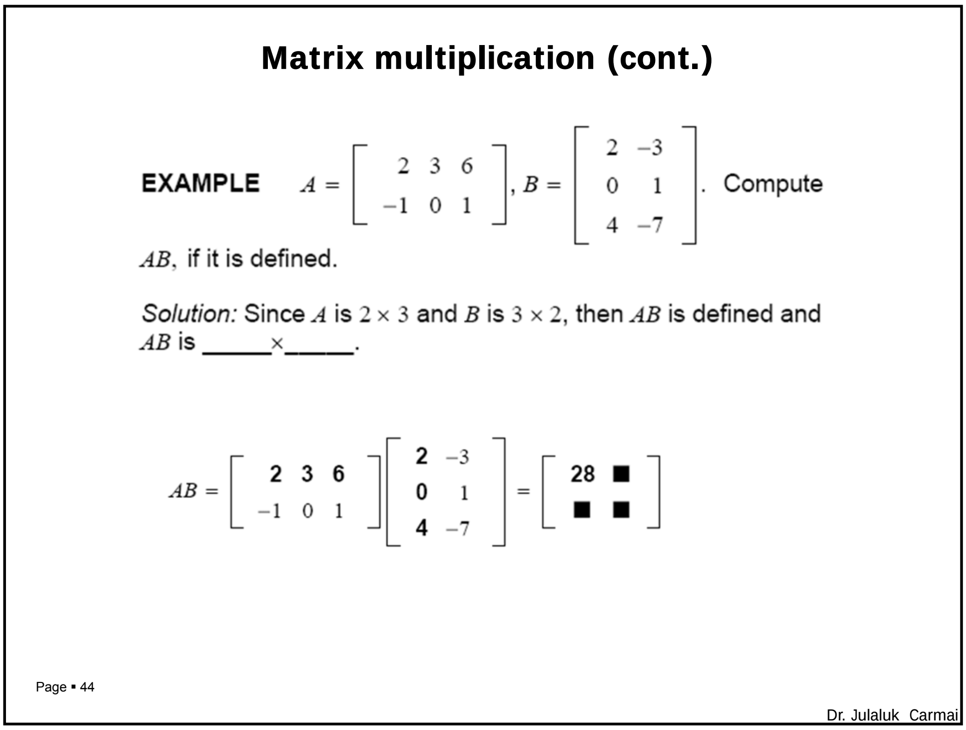
($\boldsymbol{AB}$)$_{ij}$ = row i of *A* times column j of ***B***

This single entry is the inner product of the two vectors.

Matrix multiplication(cont.)

Example

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$$

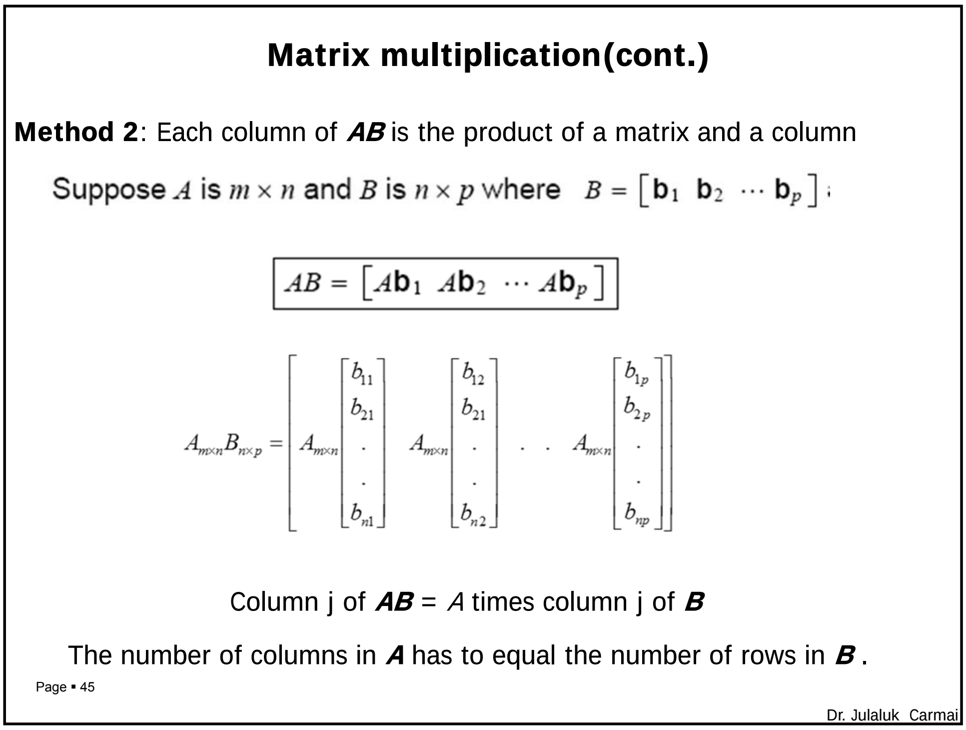


Matrix multiplication (cont.)

EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is _____×_____.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$



Matrix multiplication(cont.)

Method 2: Each column of ***AB*** is the product of a matrix and a column

Suppose A is $m \times n$ and B is $n \times p$ where $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

$$A_{m\times n}B_{n\times p} = \left[A_{m\times n}\begin{bmatrix} b_{11} \\ b_{21} \\ . \\ . \\ b_{n1} \end{bmatrix} \quad A_{m\times n}\begin{bmatrix} b_{12} \\ b_{21} \\ . \\ . \\ b_{n2} \end{bmatrix} \quad . \quad . \quad A_{m\times n}\begin{bmatrix} b_{1p} \\ b_{2p} \\ . \\ . \\ b_{np} \end{bmatrix} \right]$$

Column j of ***AB*** = *A* times column j of ***B***

The number of columns in ***A*** has to equal the number of rows in ***B*** .

Matrix multiplication(cont.)

Example $\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 3 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} =$

Example

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} =$$

Matrix multiplication(cont.)

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

Matrix multiplication(cont.)

EXAMPLE: If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \\ & \end{bmatrix}$$

Which is ____________

$$BA \text{ would be } \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Matrix multiplication(cont.)

Method 3: Each row of ***AB*** is the product of a row and a matrix

row i of ***AB*** = row i of ***A*** times ***B***

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} =$$

Properties of matrix multiplication

THEOREM 2

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left - distributive law)

c. $(B + C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$
for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

WARNINGS

Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

1. It is not the case that AB always equal BA.
2. Even if $AB = AC$, then B may not equal C.
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$.

---

1. $$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

$AB =$

$BA =$

2. $$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$B \neq C$ but $AB = AC$

3. $$A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$AB = 0$

Proof: Associative law

1. A(BC) = (AB)C

Assume B is of size m×n ; B is n×p and C is p×q

Write
$A = [a_{ij}]$
$B = [b_{ij}]$
$C = [c_{ij}]$
$(BC) = [d_{ij}]$

The (ij)-entry of A(BC) is

$$\sum_{k=1}^{n} a_{ik} d_{kj}$$

The (i,j)-entry of (BC) is

$$(BC)_{kj} = d_{kj} = \sum_{l=1}^{p} b_{kl} c_{lj}$$

So the (i,j)-entry of A(BC) is

$$\sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{p} b_{kl} c_{lj} \right) = \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik} b_{kl} c_{lj}$$

Proof associative law (cont.)

The (i,j)-entry of (AB) is

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

So the (i,j)-entry of (AB)C is

$$\sum_{l=1}^{p} \sum_{k=1}^{n} a_{ik} b_{kl} c_{lj}$$

A(BC) = (AB)C

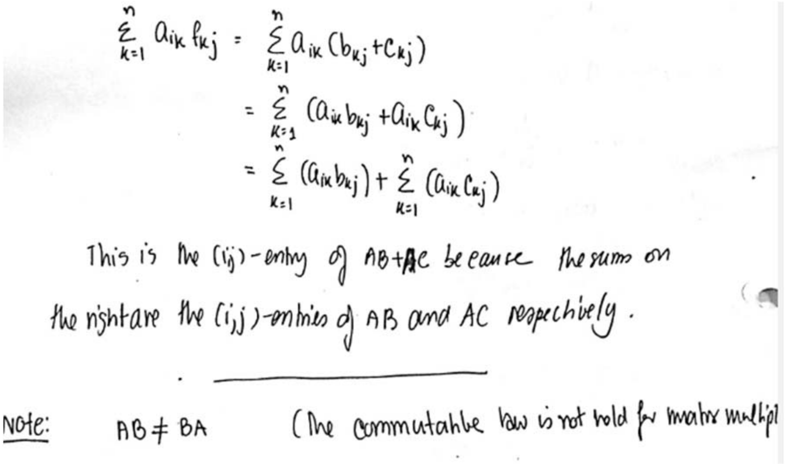
Poof: Left distributive law

A(B+C) = AB+AC (Left-distributive law)

Assume C is n×p ; A is m×n and B is n×p

Write
$A = [a_{ij}]$
$B = [b_{ij}]$
$C = [c_{ij}]$
$(B+C) = [b_{ij} + c_{ij}] = [f_{ij}]$

Poof: Left distributive law (cont.)



$$\sum_{k=1}^{n} a_{ik} f_{kj} = \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj})$$
$$= \sum_{k=1}^{n} (a_{ik}b_{kj} + a_{ik}c_{kj})$$
$$= \sum_{k=1}^{n} (a_{ik}b_{kj}) + \sum_{k=1}^{n} (a_{ik}c_{kj})$$

This is the (ij)-entry of AB+AC because the sums on the right are the (i,j)-entries of AB and AC respectively.

<u>Note:</u> $AB \neq BA$ (The commutable law is not hold for matrix multipl

Matrix transposition

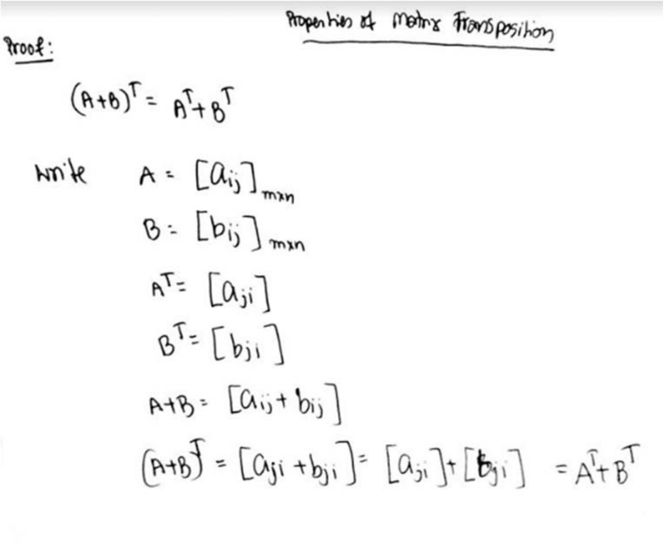
If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$$

$$\left(A^T\right)_{ij} = A_{ji}$$

Proof:



Properties of Matrix Transposition

Proof:

$(A+B)^T = A^T + B^T$

Write $A = [a_{ij}]_{m\times n}$

$B = [b_{ij}]_{m\times n}$

$A^T = [a_{ji}]$

$B^T = [b_{ji}]$

$A+B = [a_{ij} + b_{ij}]$

$(A+B)^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^T + B^T$

Matrix transposition

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute

AB, $(AB)^T$, A^TB^T and B^TA^T.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

$$\boxed{(AB)^T = \quad}$$

Properties of Matrix Transposition

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^TA^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ ________.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T$$

$$= C^T(\qquad) = \text{\_\_\_\_\_\_\_\_}.$$

Matrix Notation

Matrix Notation

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

m×n matrix is a rectangular array of numbers with m rows and n column

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

(augmented matrix)

↓

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

↓

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

Using a Matrix and Vectors to Depict a system of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = b_1$$

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = b_2$$

Coefficient matrix *A*

Constant vector *b*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

An augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \end{bmatrix} = [\underline{\mathbf{A}}|\underline{\mathbf{b}}]$$

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots a_{3n}x_n = b_3 \\ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ . \qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots a_{nn}x_n = b_n \end{array}$$

A system of linear equations can be written in a matrix equation form

$$\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

The whole idea of linear algebra is to solve

$$\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

A system of linear equations can be written in matrix form

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$$\begin{bmatrix} 6 & 1 & 1 \\ 5 & 1 & 2 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix} \quad \text{or} \quad \left[\begin{array}{ccc|c} 6 & 1 & 1 & 6 \\ 5 & 1 & 2 & 4 \\ 4 & 1 & -1 & -2 \end{array}\right]$$

Augmented matrix

A solution of the system of algebraic equations

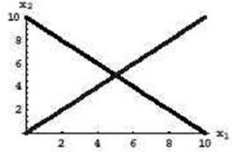
⇩

A list of numbers that makes each equation true

Possible solutions of a system of linear equations

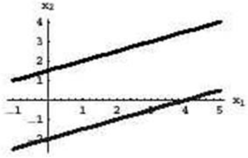
A system of linear equations has either

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$



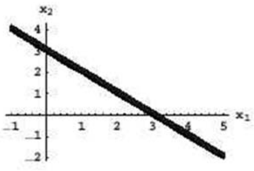
one unique solution

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$



no solution

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$



infinitely many solutions

Strategy for solving a linear system

Replace one system with an equivalent system (one with the same solution set) that is easier to solve.

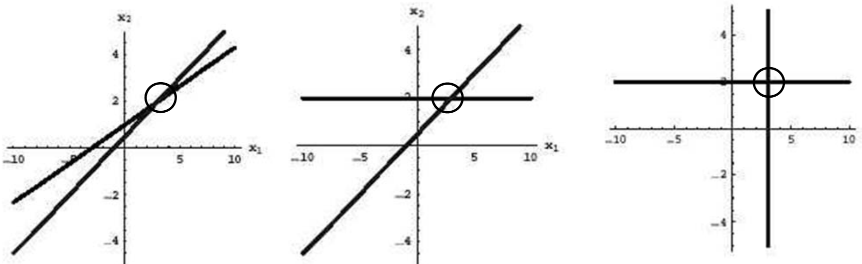
example

a)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

b)
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

c)
$$\begin{aligned} x_1 \qquad &= 3 \\ x_2 &= 2 \end{aligned}$$

Equivalent systems



$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \qquad \begin{aligned} x_1 \qquad &= 3 \\ x_2 &= 2 \end{aligned}$$

a) b) c)

Solving a linear system

Elementary Row Operations:

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Note: **Row equivalent matrices:** Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

solving a linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

solving a linear system

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Check: Is $(29, 16, 3)$ a solution of the *original* system?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$\begin{aligned} (29) - 2(16) + 3 &= 29 - 32 + 3 &&= 0 \\ 2(16) - 8(3) &= 32 - 24 &&= 8 \\ -4(29) + 5(16) + 9(3) &= -116 + 80 + 27 &&= -9 \end{aligned}$$

Row operations are reversible.

If the augmented matrices of two linear systems are row equivalent ,then the two systems have the same solution sets.

Example

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$