Chapter 6: Inner Product Spaces

1 Inner Products

1.1 inner product or dot product of $\mathbb{R}^n$

Definition 1.1. Euclidean inner product (or the standard inner product or dot product) on $\mathbb{R}^n$ is an inner product of two vectors $\mathbf{u} = [u_1, u_2, \ldots, u_n]^T$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]^T$ in $\mathbb{R}^n$ defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

The inner product $\langle \cdot, \cdot \rangle$ satisfies the following properties:

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Additivity axiom]
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{v}, \mathbf{u} \rangle$ [Homogeneity axiom]
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$ [Positivity axiom]

With the dot product we have geometric concepts such as the length of a vector, the angle between two vectors, orthogonality, etc. We shall push these concepts to abstract vector spaces so that geometric concepts can be applied to describe abstract vectors.

1.2 General Inner Product

This section aims to extend the notion of a *dot product* to general real vector spaces.

Definition 1.2. An inner product on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Symmetry axiom]
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ [Additivity axiom]
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{v}, \mathbf{u} \rangle$ [Homogeneity axiom]
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$ [Positivity axiom]

A real vector space with an inner product is called a **real inner product space**.

Definition 1.3 (Norm)**.** Let V be a real inner product space. The **norm** (or **length**) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

The distance between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(u, v) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

The following theorem provides some properties of norms and distances in real inner product spaces.

Theorem 1.1. Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, and k is a scalar. Then

(a) $\|\mathbf{v}\| > 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$

(b) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

(d) $d(\mathbf{u}, \mathbf{v}) \geq 0$ and $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

Example 1.1. Let $\mathbf{u} = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in $\mathbb{R}^2$. Verify that the weighted Euclidean inner product defined by

$$\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$$

satisfies the four inner product axioms.

Example 1.2. Let $\mathbf{u} = [1, 1]^T$ and $\mathbf{v} = [0, 1]^T$.
(a) Find the standard Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle$.
(b) Find weighted Euclidean inner product defined by $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$

1.3 Inner Products Generated by Matrices

The Euclidean inner product and the weighted Euclidean inner products are special cases of a general class of inner products on $\mathbb{R}^n$ called matrix inner products.

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$ that are expressed in column form, and let $\mathbf{A}$ be an invertible $n \times n$ matrix. The inner product on $\mathbb{R}^n$ generated by $\mathbf{A}$ is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{Au} \cdot \mathbf{Av}$$

or

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}\mathbf{A}^T\mathbf{A}\mathbf{v}$$

It can be shown that if $\mathbf{u} \cdot \mathbf{v}$ is the Euclidean inner product on $\mathbb{R}^n$, then the formula $\langle u, v \rangle = \mathbf{Au} \cdot \mathbf{Av}$ also defines an inner product.

Example 1.3. Consider the weighted Euclidean inner product of two vectors $\mathbf{u} = [u_1, u_2, \ldots, u_n]^T$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]^T$ in $\mathbb{R}^n$

$$\langle u, v \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n$$

Determine $\mathbf{A}$ that generate this weighted Euclidean inner product.

1.4 The Standard Inner Product in M_{nn}

Example 1.4. Let $\mathbf{u} = U$ and $\mathbf{v} = V$ be matrices in the vector space M_{nn}.
(a) Show that the inner product defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = tr(U^T V)$$

defines an inner product on M_{nn}. Note this is called the **standard inner product** on M_{nn}. (Exercise)
(b) Let $U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$. Find $\langle \mathbf{u}, \mathbf{v} \rangle$ defined in (a) and the corresponding norm $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

1.5 Inner Product on P_n

Let $\mathbf{p} = p(x) = a_0 + a_1x + \cdots + a_nx_n$ and $\mathbf{q} = q(x) = b_0 + b_1x + \cdots + b_nx_n$ be polynomials in P_n.
(i) The **standard inner product on** P_n is defined as

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0b_0 + a_1b_1 + ... + a_nb_n$$

The norm of a polynomial p relative to this inner product is

$$\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} =$$

(ii) **The Evaluation Inner Product on** P_n
Suppose x_0, x_1, ... , x_n are distinct real numbers (called sample points), we can define an inner product called **the evaluation inner product at** $x_0, x_1, ..., x_n$ as follow:

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + ... + p(x_n)q(x_n)$$

Example 1.5. The evaluation inner product
(a) Show that **the evaluation inner product at** $x_0, x_1, ..., x_n$ defined above satisfies all axioms for inner product. (Exercise)
(b) Let P_2 have the *evaluation inner product* at the points $x_0 = 2$, $x_1 = 0$, and $x_2 = 2$. Compute $\langle \mathbf{p}, \mathbf{q} \rangle$ and $\|\mathbf{p}\|$ for the polynomials $\mathbf{p} = p(x) = x^2$ and $\mathbf{q} = q(x) = 1 + x$.

1.6 An Integral Inner Product on $C[a, b]$

Let $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ be two functions in $C[a, b]$ and define

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)dx.$$

and the corresponding norm is

$$\|\mathbf{f}\| =$$

The inner product formula above defines an inner product on C[a, b], i.e. the four inner product axioms are satisfied for functions $= \mathbf{f}(\mathbf{x})$, $\mathbf{g} = g(x)$, and $\mathbf{h} = h(x)$ in $C[a, b]$:

Example 1.6. Let $\mathbf{f} = f(x) = x^2$ and $\mathbf{g} = g(x) = \frac{1}{x}$ be two functions in $C[0, 1]$. Find the integral inner product on $C[0, 1]$ of $\mathbf{f}$ and $\mathbf{g}$.

Example 1.7. Prove the following algebraic properties of inner products in the following theorem.

Theorem 1.2. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a real inner product space V, and k be a scalar. Then

(a) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(b) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(c) $\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$

(d) $\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$

(e) $k\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle$

Proof (a) and (b):

2 Angle and Orthogonality in Inner Product Spaces

Recall that the dot product between vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\mathbf{v}\| \cos(\theta),$$

where θ is the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$. So,

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right).$$

The following generalization will enable us to define the angle between two vectors in any real inner product space.

Theorem 2.1. Cauchy-Schwarz Inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space V, then

$$\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\|\mathbf{v}\|$$

Example 2.1. (Exercise) Use Cauchy-Schwarz inequality to show that

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1.$$

Definition 2.1. The **angle** between vectors non-zero $\mathbf{u}$ and $\mathbf{v}$ in a real inner product space is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right), \quad 0 \le \theta \le \pi$$

Example 2.2. Let M_{22} have the standard inner product. Find the cosine of the angle between the vectors $\mathbf{u} = U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{v} = V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$.

Theorem 2.2. Triangle Inequality
Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in a real inner product space V. Then

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

2.1 Orthogonality

Definition 2.2. Orthogonality Two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V called orthogonal if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

Example 2.3. Let $\mathbf{u} = [1, 1]^T$ and $\mathbf{v} = [1, -1]^T$.
Determine if the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal with respect to the following inner products.
(a) The Euclidean inner product on $\mathbb{R}^2$
(b) The weighted Euclidean inner product $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$

Example 2.4. Let $U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$. Determine if the vectors U and V are orthogonal with respect to the standard inner product in M_{22}.

Example 2.5. Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

(a) Determine if the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal with respect to the above inner product.
(b) Determine if the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^3$ are orthogonal with respect to the above inner product.

Theorem 2.3. (**Generalized Theorem of Pythagoras**) Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in a real inner product space. Then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Example 2.6. Let P_2 have the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Let $\mathbf{p} = x$ and $\mathbf{q} = x^2$. Find $\|\mathbf{p}+\mathbf{q}\|^2$.
(I) Direct approach:

(II) Use Generalized Theorem of Pythagoras:
Note, we already have:

Definition 2.3. If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^\perp$.

Theorem 2.4. If W is a subspace of a real inner product space V, then
(a) $W^\perp$ is a subspace of V .
(b) $W \cap W^\perp = \{\mathbf{0}\}$.

Proof:

Remark:If W is a subspace of a real finite-dimensional inner product space V, then the orthogonal complement of $W^\perp$ is W;i.e. $\boxed{(W^\perp)^\perp = W.}$

Example 2.7. . Let W be the subspace of $\mathbb{R}^6$ spanned by the vectors $\mathbf{w}_1 = [1,3,-2,0,2,0]^T, \mathbf{w}_2 = [2,6,-5,-2,4,-3]^T, \mathbf{w}_3 = [0,0,5,10,0,15]^T, \mathbf{w}_4 = [2,6,0,8,4,18]^T$
Find a basis for the orthogonal complement of W.