SUBJECT: EE320 - Unconstrained Optimization NO: 1 DATE: / /

3-independent-variable case. (and n-var).

- Given $z = f(x_1, x_2, x_3)$

$\Rightarrow dz = f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3$, where $f_i = \frac{\partial z}{\partial x_i}$; $i = 1, 2, 3$.

FONC: $dz = 0 \iff f_1 = f_2 = f_3 = 0$.

SOSC: For maximum, $d^2z < 0$
For minimum, $d^2z > 0$.

$$d^2z = d(dz) = \frac{\partial(dz)}{\partial x_1}\cdot dx_1 + \frac{\partial(dz)}{\partial x_2}\cdot dx_2 + \frac{\partial(dz)}{\partial x_3}\cdot dx_3$$

$$= \frac{\partial}{\partial x_1}(f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3)\,dx_1 + \frac{\partial}{\partial x_2}(f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3)\,dx_2 + \frac{\partial}{\partial x_3}(f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3)\,dx_3$$

$$= (f_{11}\,dx_1 + f_{12}\,dx_2 + f_{13}\,dx_3)\,dx_1 + (f_{21}\,dx_1 + f_{22}\,dx_2 + f_{23}\,dx_3)\,dx_2 + (f_{31}\,dx_1 + f_{32}\,dx_2 + f_{33}\,dx_3)\,dx_3$$

$$= f_{11}\,dx_1^2 + f_{12}\,dx_2dx_1 + f_{13}\,dx_3dx_1 + f_{21}\,dx_1dx_2 + f_{22}\,dx_2^2 + f_{23}\,dx_3dx_2 + f_{31}\,dx_1dx_3 + f_{32}\,dx_2dx_3 + f_{33}\,dx_3^3$$

$$\therefore d^2z = f_{11}(dx_1)^2 + f_{22}(dx_2)^2 + f_{33}(dx_3)^2 + 2f_{12}\,dx_1dx_2 + 2f_{13}\,dx_1dx_3 + 2f_{23}\,dx_2dx_3$$

In matrix form:

$$d^2z = \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix} \underbrace{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}}_{\text{Hessian matrix: } H_{3\times 3}} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

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$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

⇒ 1st leading principal minor : $|H_1| = f_{11}$

2nd " —————— " : $|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11}f_{22} - (f_{12})^2$

3rd " —————— " : $|H_3| = |H|$.

∴ SOSC : (for $z = f(x_1, x_2, x_3)$).

① For maximum, we need:

$d^2z < 0 \Leftrightarrow |H_1| < 0 \;\&\; |H_2| > 0 \;\&\; |H| < 0.$

② For minimum, we need:

$d^2z > 0 \Leftrightarrow |H_1| > 0 \;\&\; |H_2| > 0 \;\&\; |H| > 0$

■ n-variable case

Given $z = f(x_1, x_2, \dots, x_n)$.

FONC: $dz = 0 \Leftrightarrow f_1 = f_2 = \dots = f_n = 0.$

SOSC:

① Max : $|H_i|$ alternates in sign, where $|H_1| < 0$.
(ie. $|H_1| < 0$, $|H_2| > 0$, $|H_3| < 0$, ...).

② Min : $|H_i| > 0$, for all $i = 1, \dots, n$.

where $H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$