EE320 Chapter 10
Integration and Its Application

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1 Terminology

Integration = anti-derivative or inverse of differentiation

$\boxed{y = F(x)} \Rightarrow F(x) = \int F'(x)dx + c$

$\frac{dy}{dx} = F'(x)$

Indefinite integral

If we integrate f(x) where values of x are not given, we have to integrate without a limit. (i.e. to find indefinite integral)

A symbol for integrating a function f(x) is

$$\int f(x)dx = F(x) + c \Leftrightarrow F'(x) = f(x)$$

where

$\int$	is	integral sign
f(x)	is	integrand
c	is	constant of integration
dx	indicates	the variable involved in the integration

<u>Note</u> function does not have a unique integral $c_1, c_2, \dots$

ex. Given a function $\boxed{f(x) = 3x^2}$, a possible result from integration of $f(x)$ is that: F(x) = $x^3 + 1$, $x^3 + 7$, $x^3 + c$

$\int f(x)dx = x^3+1 , x^3+7 , x^3+c$

$\frac{d}{dx} x^3+1 = 3x^2 = f(x)$

$\frac{d}{dx} x^3+7 = 3x^2 = f(x)$

$\frac{d}{dx} x^3+c = 3x^2 = f(x)$

2 Basic Rules for Integration

I) $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$, $(n \neq -1)$

II) $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\Rightarrow \int a^{bx} dx = \frac{1}{b \ln a}a^{bx} + c$

$\Rightarrow \int f(x)e^{f(x)}dx = e^{f(x)} + c$

$\Rightarrow \int \frac{f'(x)}{f(x)}dx = ln|f(x)| + c, f(x) \neq 0$

III) $\int \frac{1}{x}dx = ln|x| + c$

Addition/Substraction rule : $f(x), g(x) \Rightarrow \int f(x)+g(x)\,dx$

IV) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

add/substract first & then integrate   separate integration

V) $\int af(x)dx = a \int f(x)dx$, a = constant

$\int af(x)\,dx = a\int f(x)\,dx$.

$$\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{1}{-3+1}x^{(-3+1)} + C \qquad n=-3$$
$$= -\frac{1}{2}x^{-2} + C$$

ex.

1. $\int \frac{1}{x^3}dx \quad = \quad -\frac{1}{2x^2} + c$

2. $\int \sqrt{x\sqrt{x\sqrt{x}}}dx \quad = \quad \frac{8}{15}x^{\frac{15}{8}} + c$

3. $\int (3x^4 + 5x^2 - 2)dx \quad = \quad \frac{3}{5}x^5 + \frac{5x^3}{3} - 2x + c$ ✓

4. $\int e^{3x} - e^{2x} + e^x dx \quad = \quad \frac{e^{3x}}{3} - \frac{e^{2x}}{2} + e^x + c$ ✓

5. $\int \frac{(y-2)^2}{\sqrt{y}}dy \quad = \quad \frac{2}{5}y^{\frac{5}{2}} - \frac{8}{3}y^{\frac{3}{2}} + 8y^{\frac{1}{2}} + c$ ✓

6. $\int 2^x dx \quad = \quad \cancel{\frac{1}{\ln 2}2^x} + c \quad \frac{1}{\ln 2} \cdot 2^x + C$

$$\int \sqrt{x\sqrt{x\sqrt{x}}}\,dx = \int \sqrt{x\sqrt{x \cdot x^{\frac{1}{2}}}}\,dx$$
$$= \int \sqrt{x\sqrt{x^{3/2}}}\,dx$$
$$= \int \sqrt{x(x^{3/2})^{\frac{1}{2}}}\,dx$$
$$= \int \sqrt{x \cdot x^{3/4}}\,dx$$
$$= \int \sqrt{x^{7/4}}\,dx$$
$$= \int x^{7/8}\,dx$$

VI) $\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c$

ex. $\int (ax+b)^P dx$ let $u = ax+b \Rightarrow du = a\,dx \Leftrightarrow dx = du/a$

$$\left(\begin{matrix} \int x^p dx \\ \int (ax+b)^p dx \end{matrix} \right)$$

a, b, p = constant
x, u = variables

$$\int (ax+b)^p dx = \int u^p \frac{du}{a}$$
$$= \frac{u^{p+1}}{a(p+1)} + C$$
$$= \frac{(ax+b)^{p+1}}{a(p+1)} + C \quad \#$$

VII) $\int v\ du = uv - \int u\ dv$ "Integration by parts"

ex. $\int xe^x dx$

$$\Rightarrow \text{let } v = x \Rightarrow dv = dx$$
$$u = e^x \Rightarrow du = e^x dx$$

Initial-Value Theorem

$$\boxed{\int f(x)dx = F(x) + c}$$

If we have an initial condition, we can determine the value of c.

ex. 1. Find F(x) if $F'(x) = \frac{1}{2} - 2x$ and $\boxed{F(0) = \frac{1}{2}}$

$\int \frac{1}{2} - 2xdx = \frac{1}{2}x - \frac{2x^2}{2} + C = \frac{x}{2} - x^2 + C = F(x)$

$F(0) = \frac{0}{2} - 0^2 + C = \frac{1}{2} \Rightarrow C = \frac{1}{2} \Rightarrow F(x) = \frac{x}{2} - x^2 + \frac{1}{2}.$

2. Find F(x) if $F'(x) = x(1 - x^2)$ and $F(0) = \frac{5}{12}$

$\int x(1 - x^2)dx = \frac{x^2}{2} - \frac{x^4}{4} + C = F(x)$

$F(0) = \frac{0^2}{2} - \frac{0^4}{4} + C = \frac{5}{12} \Rightarrow C = \frac{5}{12} \Rightarrow F(x) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{5}{12}$

integrate
Total Revenue $\rightleftarrows$ Marginal Revenue
differentiate

Application 1: Derivation of TR from MR

$\boxed{\text{TR} = \int \text{MR(Q) dQ}}$

ex. Given MR = 10 - Q, find TR.

$\text{TR} = \int 10 - Q dQ = 10Q - \frac{Q^2}{2} + C$

✓ Suppose at Q = 0 , TR = 0 $\Rightarrow$ c = 0 $\therefore$ $\boxed{\text{TR} = 10Q - \frac{Q^2}{2}}$

$TR(Q=0) = \cancel{10(0)} - \cancel{\frac{0^2}{2}} + C = 0$

$C = 0$

Application 2: Derivation of TC from MC

integrate
TC $\rightleftarrows$ MC
differentiate

$\boxed{\text{TC}} = \int \text{MC(Q) dQ}$

ex. Given MC $= 2e^{0.2Q}$, $C_F = 90$, find TC.

$\text{TC(Q)} = \int 2 \cdot e^{0.2Q} dQ = \frac{2}{0.2} e^{0.2Q} + C = 10 e^{0.2Q} + C$

$\boxed{\text{TC(0)}} = 90 = 10 \cdot e^{0.2 \cdot (0)} + C \Rightarrow C = 80$

$\Rightarrow \boxed{TC = 10 \cdot e^{0.2Q} + 80.}$

$\pi(Q) = TR - TC$

$\pi'(Q) = MR - MC$

Application 3: Derivation of Profit function from MR-MC

$\Pi'(Q)$ = Marginal profit = MR-MC

$\Pi = \int \Pi'(Q)dQ = \int MR\, dQ - \int MC\, dQ$

ex. Given MR = 50 - 2Q , MC = 10+Q, find total profit when Q =10. Assume there is no fixed cost: $\Pi(10)$ = ?

$\Pi'(Q) = (50-2Q) - (10+Q)$

$\int \Pi'(Q)$ dQ = $\int 40 - 3Q\, dQ = 40Q - \frac{3Q^2}{2} + C$

If Q = 0 , TR = 0 , TC = 0 $\quad\Rightarrow\quad \Pi(Q) = 0 \quad\Rightarrow\quad c = 0$

$\therefore \boxed{\Pi(Q) = 40Q - \frac{3Q^2}{2}} \Rightarrow \pi(10) = 400 - \frac{300}{2} = 250.$

Application 4: Derivation of Utility function from MU

U(x)= $\int MU(x)dx$

ex. Given MU(x) = $\frac{5}{3\sqrt{x}}$, find U(x).

U(x) = $\int \frac{5}{3\sqrt{x}} dx = \frac{10}{3}\sqrt{x} + C$.

$= \int \frac{5}{3} x^{-\frac{1}{2}} dx$

Application 5: Derivation of Consumption/Saving function from marginal propensity function

ex. Given marginal propensity to save: $S'(Y) = 0.3 - 0.1Y^{-\frac{1}{2}}$ and $S(81) = 0$, find saving and consumption function

S(Y) = $\int 0.3 - 0.1Y^{-\frac{1}{2}}\,dY$

S(Y) = $0.3Y - 0.2\sqrt{Y} + C$

S(81) = $0.3(81) - 0.2\sqrt{81} + C = 0. \Rightarrow C = -22.5$

∴ S(Y) = $0.3Y - 0.2\sqrt{Y} - 22.5$

For C(Y), $C = Y - S =$ $Y - [0.3Y - 0.2\sqrt{Y} - 22.5] = 0.7Y + 0.2\sqrt{Y} + 22.5$

3 Definite Integrals

$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a)$

ex. $\int_0^2 5x^2\,\mathrm{d}x$ = $\frac{5x^3}{3}\Big|_0^2 = \frac{5(2)^3}{3} - \frac{5(0)^3}{3} = \frac{40}{3}$

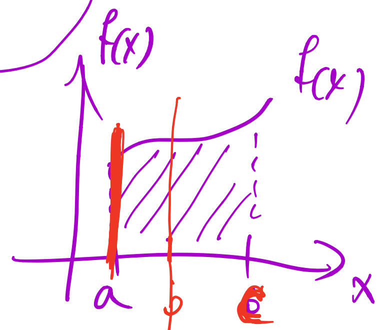
$\int_0^1 ax^b\,\mathrm{d}x$ = $\frac{ax^{b+1}}{b+1}\Big|_0^1 = \frac{a(1)^{b+1}}{b+1} - \frac{a(0)^{b+1}}{b+1} = \frac{a}{b+1}$

a, b = constant

Area and definite integral

The area under the graph of a continuous and nonnegative function f(x) over the interval [a,b] is $\int_a^b f(x)\,\mathrm{d}x$ or

$$\text{Area } A = \lim_{\Delta x \to 0} {}^{b}_{a}[f(x) - \Delta x] = \int_a^b f(x)\,\mathrm{d}x$$



Properties of Definite Integrals

1. $\int_a^b f(x)\,\mathrm{d}x \quad = \quad \int_b^a -f(x)\,\mathrm{d}x$
2. $\int_a^a f(x)\,\mathrm{d}x \quad = \quad 0$
3. $\int_a^c f(x)\,\mathrm{d}x \quad = \quad \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$, $(a<b<c)$
4. $\int_a^b -f(x)\,\mathrm{d}x \quad = \quad -\int_a^b f(x)\,\mathrm{d}x$
5. $\int_a^b \alpha f(x)\,\mathrm{d}x \quad = \quad \alpha \int_a^b f(x)\,\mathrm{d}x$
6. $\int_a^b [f(x) + g(x)]\,\mathrm{d}x \quad = \quad \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
7. $\int_{x=a}^{x=b} v\,\mathrm{d}u \quad = \quad uv\,\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,\mathrm{d}u$ by ~~parts~~

ex.

1. $\int_0^5 (x+x^2)\mathrm{d}x \quad = \quad 325/6$
2. $\int_2^4 x^2(\frac{1}{3}x^3+1)\,\mathrm{d}x \quad = \quad 1184/9$
3. $\int_{-2}^2 e^x - e^{-x}\,\mathrm{d}x \quad = \quad 0$
4. $\int_e^6 (\frac{1}{x}+\frac{1}{1+x})\,\mathrm{d}x \quad = \quad (\ln 6 + \ln 7) - (1 - \ln(e+1))$
5. $\int_{-2}^3 |x+1|\,\mathrm{d}x \quad = \quad 8.5$

Application: Capital Formation and Investment Functions

Definition : K(t) = capital stock at t

$\frac{d}{dt}K(t)$ = rate of capital formation

$I(t)$ = rate of net investment flow at t

$$\boxed{\frac{d}{dt}K(t) \equiv I(t)} \quad \Rightarrow \quad \int I(t)dt = \int \frac{dK}{dt}dt = K(t)$$

Gross investment $\equiv I_g(t) = I(t) + \delta K(t)$ $\quad \delta$ = depreciation rate.

Capital formation during a time interval [a,b] $= \int_a^b I(t)\,\mathrm{d}t = K(t)\,|_a^b$

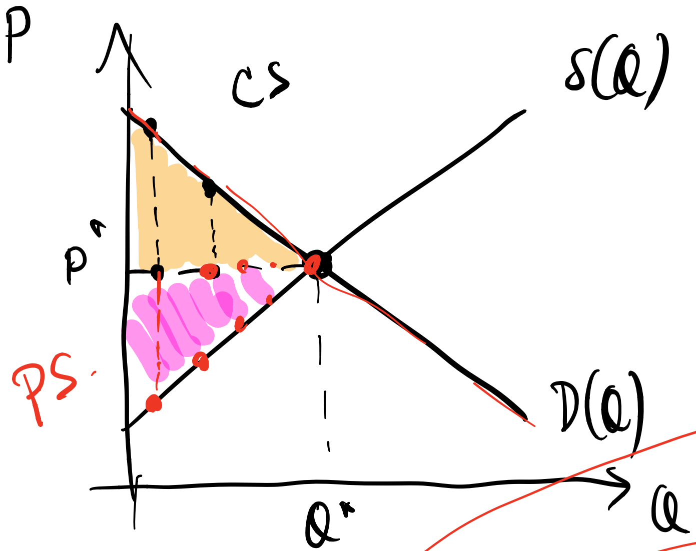
ex. Suppose net investment flow is I(t) $= 3t^{\frac{1}{2}}$ and the initial capital stock at t = 0 is K = 25. What is K(t) during [1,4]?

$K(t=0) = 25$, $K(t)\Big|_1^4$

$K(t) = \int I(t)\,dt = \int 3t^{\frac{1}{2}}\,dt = 2t^{3/2} + C$ no need to find c in this case

$K(4) - K(1) = 2(4)^{3/2} + \cancel{C} - 2(1)^{3/2} - \cancel{C} = 14.$

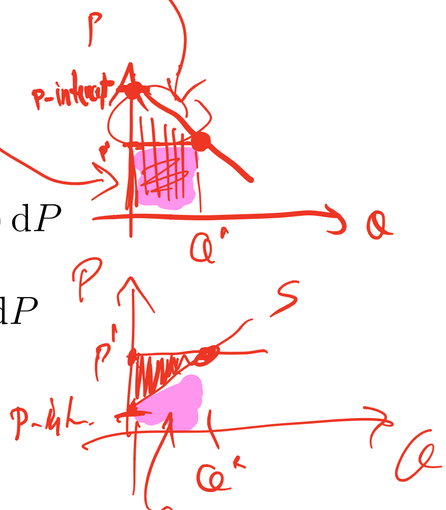
Application: Consumer and Producer Surpluses



$D(Q)$: $P = a + bQ$.

$Q^d(P)$: $Q = \frac{P-a}{b}$

$D(Q) \Rightarrow Q^d(P)$



$\int S(Q)\, dQ$

$$\text{CS} = \int_0^{Q^*} D(Q) - P^*\, \mathrm{d}Q = \int_{P^*}^{P-\text{intercept}} Q^d(P)\, \mathrm{d}P$$

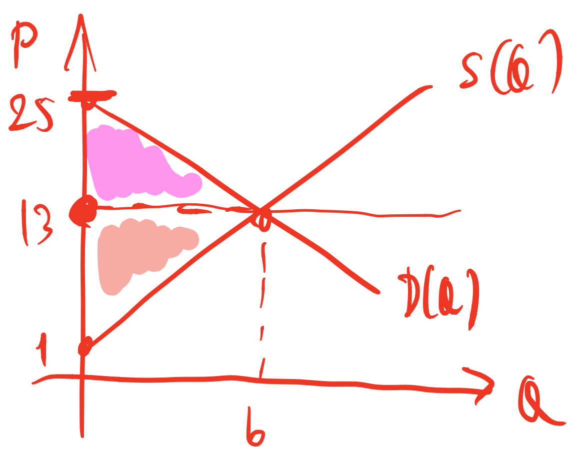
$$\text{PS} = \int_0^{Q^*} P^* - S(Q)\, \mathrm{d}Q = \int_{P-\text{intercept}}^{P^*} S(Q)\, \mathrm{d}P$$

ex. $\boxed{\text{S(P)} = -\frac{1}{2} + \frac{1}{2}P}$

$\boxed{\text{D(P)} = \frac{25}{2} - \frac{1}{2}P}$

Find PS , CS and total welfare PS+CS

$\Rightarrow P^* = 13,\ Q^* = 6$



$Q^s = S(P) = -\frac{1}{2} + \frac{1}{2}P$, $Q^d = \frac{25}{2} - \frac{1}{2}P$

$\Rightarrow P = 1 + 2Q^s$

$P = 25 - 2Q^d$

① $\text{CS} = \int_0^6 25 - 2Q^d - 13\, dQ$

$= 36$

$\text{PS} = \int_0^6 13 - (1 + 2Q^s)\, dQ$

$= 36$

Total welfare $= 36 + 36 = 72$

$$Q^d = \frac{25}{2} - \frac{1}{2}P \Leftrightarrow P = 25 - 2Q^d$$

② $CS = \int_{13}^{25} \frac{25}{2} - \frac{1}{2}P \, dP = 36.$

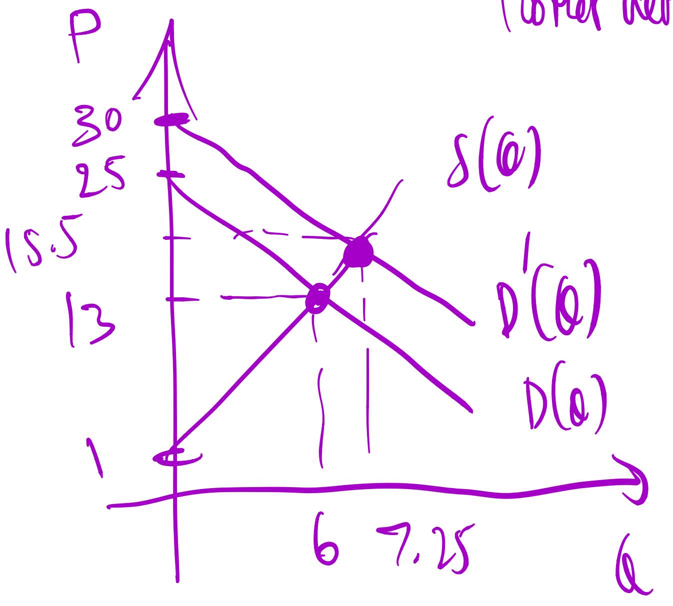
$PS = \int_{1}^{13} -\frac{1}{2} + \frac{1}{2}P \, dP = 36$

CS + PS = 72.

$$Q^S = -\frac{1}{2} + \frac{1}{2}P \Leftrightarrow P = 1 + 2Q^S$$

ex. If the demand changes to $\boxed{D(P) = \frac{30}{2} - \frac{1}{2}P}$, ΔCS and ΔPS?

former demand : $D(P) = \frac{25}{2} - \frac{1}{2}P$



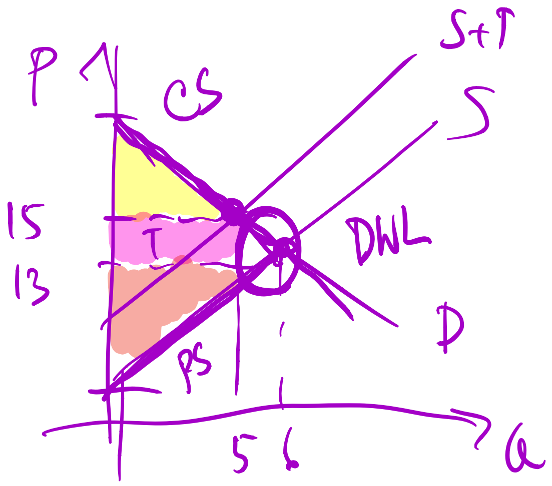
$\Rightarrow P^* = 15.5,\ Q^* = 7.25$

$CS' = 52.5625 \Rightarrow CS' - CS = 16.5625$

$PS' = 52.5625 \Rightarrow PS' - PS = 16.5625$

ex. If the demand does not change and government imposes \$ 4 per unit tax on producer instead,

$$\Rightarrow \quad P^S = P - 4 \quad \Rightarrow \quad \left.\begin{aligned} Q^S &= -\frac{1}{2} + \frac{1}{2}(P-4) \\ Q^d &= \frac{25}{2} - \frac{1}{2}P \end{aligned}\right\} \; P^* = 15, \; Q^* = 5$$



$CS'' = 25$

$PS'' = 25$

$$\left.\begin{aligned} \Delta CS &= 11 \\ \Delta PS &= 11 \end{aligned}\right\} \Delta \text{ total welfare} = 22.$$

$T = 20.$

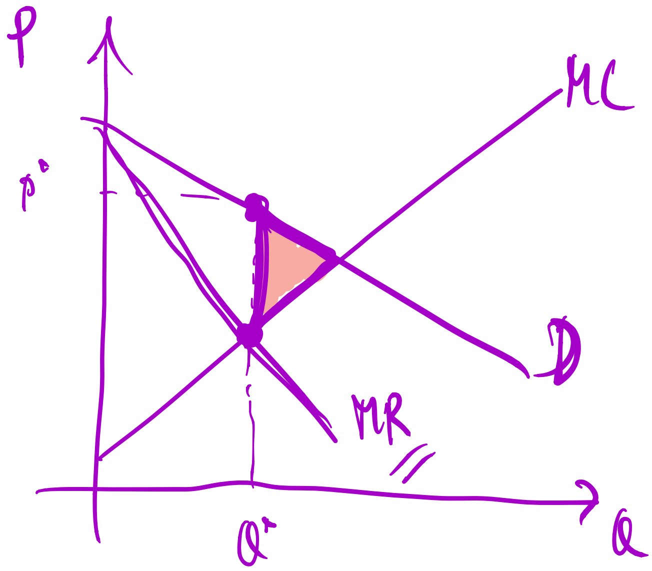
$$DWL = 2 = \Big(\underbrace{\text{Total welfare before}}_{72} - \underbrace{\text{TW after tax}}_{50 + 20}\Big)$$

72   70

Application: First-Degree Price discrimination or Perfect Price discrimination

Monopolist charges the maximum price for each unit of output sold.

ex. Suppose that a monopolist faces a demand function P = 24 -Q, and MC = 4 + 3Q. Find CS and PS at profit-maximized Q^*



$Q^* = 4.$

$p^* = 20.$

$PS = 40$

$CS = 8.$

Reference

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