

# Basics of Derivative Pricing

George Pennacchi

University of Illinois

# Introduction

- Derivative securities have cashflows that derive from another “underlying” variable, such as an asset price, interest rate, or exchange rate.
- The absence of arbitrage opportunities places restrictions on the derivative’s value relative to that of its underlying asset.
- For forward contracts, no-arbitrage considerations alone may lead to an exact pricing formula.
- For options, no-arbitrage restrictions cannot determine an exact price, but only bounds on the option’s price.
- An exact option pricing formula requires additional assumptions on the probability distribution of the underlying asset’s returns (e.g., binomial).

## Forward Contracts on Assets Paying Dividends

- Let  $F_{0\tau}$  be the date 0 forward price for exchanging one share of an underlying asset  $\tau$  periods in the future. This price is agreed to at date 0 but paid at date  $\tau > 0$  for delivery at date  $\tau$  of the asset.
- Hence, the date  $\tau > 0$  payoff to the long (*short*) party in this forward contract is  $S_\tau - F_{0\tau}$ , ( $F_{0\tau} - S_\tau$ ) where  $S_\tau$  is the date  $\tau$  spot price of one share of the underlying asset.
- The parties set  $F_{0\tau}$  to make the date 0 contract's value equal 0 (no payment at date 0).
- Let  $R_f > 1$  be the per-period risk-free return for borrowing or lending over the period from date 0 to date  $\tau$ , and let  $D$  be the date 0 present value of dividends paid by the underlying asset over the period from date 0 to date  $\tau$ .

# Forward Contract Cash Flows

- Consider a long forward contract and the trades that would exactly replicate its date  $\tau$  payoffs:

| <u>Date 0 Trade</u>             | <u>Date 0 Cashflow</u>             | <u>Date <math>\tau</math> Cashflow</u> |
|---------------------------------|------------------------------------|--|
| <b>Long Forward Contract</b>    | 0                                  | $S_\tau - F_{0\tau}$                   |
| <b>Replicating Trades</b>       |                                    |  |
| 1) Buy Asset and Sell Dividends | $-S_0 + D$                         | $S_\tau$                               |
| 2) Borrow                       | $R_f^{-\tau} F_{0\tau}$            | $-F_{0\tau}$                           |
| <i>Net Cashflow</i>             | $-S_0 + D + R_f^{-\tau} F_{0\tau}$ | $S_\tau - F_{0\tau}$                   |

- In the absence of arbitrage, the cost of the replicating trades equals the zero cost of the long position:

$$S_0 - D - R_f^{-\tau} F_{0\tau} = 0 \quad (1)$$

or

$$F_{0\tau} = (S_0 - D) R_f^\tau \quad (2)$$

## Forward Contract Replication

- If the contract had been initiated at a previous date, say date  $-1$ , at the forward price  $F_{-1\tau} = X$ , then the date 0 value (replacement cost) of the long party's payoff, say  $f_0$ , would still be the cost of replicating the two cashflows:

$$f_0 = S_0 - D - R_f^{-\tau} X \quad (3)$$

- The forward price in equation (2) did not require an assumption regarding the random distribution of the underlying asset price,  $S_\tau$ , because it was a *static* replication strategy.
- Replicating option payoffs will entail, in general, a *dynamic* replication strategy requiring distributional assumptions.

## Basic Characteristics of Option Prices

- The owner of a *call* option has the right to buy an asset in the future at a pre-agreed price, called the *exercise* or *strike* price.
- Since the option owner's payoff is always non-negative, this buyer must make an initial payment to the seller.
- A *European* option can be exercised only at the maturity of the option contract.
- Let  $S_0$  and  $S_T$  be the current and maturity date prices per share of the underlying asset,  $X$  be the exercise price, and  $c_t$  and  $p_t$  be the date  $t$  prices of European call and put options, respectively.
- Then the maturity values of European call and put options are

$$c_T = \max[S_T - X, 0] \quad (4)$$

$$p_T = \max[X - S_T, 0] \quad (5)$$

## Lower Bounds on European Option Values

- Recall that the long (*short*) party's payoff of a forward contract is  $S_T - F_{0T}$  ( $F_{0T} - S_T$ ).
- If  $F_{0T}$  is like an option's strike,  $X$ , then assuming  $X = F_{0T}$  implies the payoff of a call (*put*) option weakly dominates that of a long (*short*) forward.
- Because equation (3) is the current value of a long forward position contract, the European call's value must satisfy

$$c_0 \geq f_0 = S_0 - D - R_f^{-T} X \quad (6)$$

- Furthermore, combining  $c_0 \geq 0$  with (6) implies

$$c_0 \geq \max [S_0 - D - R_f^{-T} X, 0] \quad (7)$$

- By a similar argument,

$$p_0 \geq \max [-f_0, 0] = \max [R_f^{-T} X + D - S_0, 0] \quad (8)$$

# Put-Call Parity

- *Put-call parity* links options written on the same underlying, with the same maturity date, and exercise price.

$$c_0 + R_f^{-\tau} X + D = p_0 + S_0 \quad (9)$$

- Consider forming the following two portfolios at date 0:
  - 1 Portfolio A = a put option having value  $p_0$  and a share of the underlying asset having value  $S_0$
  - 2 Portfolio B = a call option having value  $c_0$  and a bond with initial value of  $R_f^{-\tau} X + D$

Then at date  $\tau$ , these two portfolios are worth:

- Portfolio A =  $\max[X - S_\tau, 0] + S_\tau + DR_f^\tau = \max[X, S_\tau] + DR_f^\tau$
- Portfolio B =  $\max[0, S_\tau - X] + X + DR_f^\tau = \max[X, S_\tau] + DR_f^\tau$

# American Options

- An American option is at least as valuable as its corresponding European option because of its early exercise right.
- Hence if  $C_0$  and  $P_0$ , the current values of American options, then  $C_0 \geq c_0$  and  $P_0 \geq p_0$ .
- Some American options' early exercise feature has no value.
- Consider a European call option on a non-dividend-paying asset, and recall that  $c_0 \geq S_0 - R_f^{-T}X$ .
- An American call option on the same asset exercised early is worth  $C_0 = S_0 - X < S_0 - R_f^{-T}X < c_0$ , a contradiction.
- For an American put option, selling the asset immediately and receiving  $\$X$  now may be better than receiving  $\$X$  at date  $T$  (which has a present value of  $R_f^{-T}X$ ). At exercise  $P_0 = X - S_0$  may exceed  $R_f^{-T}X + D - S_0$  if remaining dividends are small.

# Binomial Option Pricing

- The no-arbitrage assumption alone cannot determine an exact option price as a function of the underlying asset.
- However, particular distributional assumptions for the underlying asset can allow the option's payoff to be replicated by trading in the underlying asset and a risk-free asset.
- Cox, Ross, and Rubinstein (1979) developed a binomial model to value a European option on a non-dividend-paying stock.
- The model assumes that the current stock price,  $S$ , either moves up by a proportion  $u$ , or down by a proportion  $d$ , each period. The probability of an up move is  $\pi$ .

# Binomial Option Pricing cont'd

$$S \begin{cases} \nearrow & uS \text{ with probability } \pi \\ \searrow & dS \text{ with probability } 1 - \pi \end{cases} \quad (10)$$

- Let  $R_f$  be one plus the risk-free rate for the period, where in the absence of arbitrage  $d < R_f < u$ .
- Let  $c$  equal the current value of a European call option written on the stock and having a strike price of  $X$ , so that its payoff at maturity  $\tau$  equals  $\max[0, S_\tau - X]$ .
- Thus, *one period prior to maturity*:

# Binomial Option Pricing cont'd

$$\begin{array}{l}
 c \\
 \nearrow \\
 c_u \equiv \max[0, uS - X] \text{ with probability } \pi \\
 \searrow \\
 c_d \equiv \max[0, dS - X] \text{ with probability } 1 - \pi
 \end{array}
 \tag{11}$$

- To value  $c$ , consider a portfolio containing  $\Delta$  shares of stock and  $B$  of bonds so that its current value is  $\Delta S + B$ .
- This portfolio's value evolves over the period as

$$\begin{array}{l}
 \Delta S + B \\
 \nearrow \\
 \Delta uS + R_f B \text{ with probability } \pi \\
 \searrow \\
 \Delta dS + R_f B \text{ with probability } 1 - \pi
 \end{array}
 \tag{12}$$

## Binomial Option Pricing cont'd

- With two securities (bond and stock) and two states (up or down),  $\Delta$  and  $B$  can be chosen to replicate the option's payoffs:

$$\Delta uS + R_f B = c_u \quad (13)$$

$$\Delta dS + R_f B = c_d \quad (14)$$

- Solving for  $\Delta$  and  $B$  that satisfy these two equations:

$$\Delta^* = \frac{c_u - c_d}{(u - d)S} \quad (15)$$

$$B^* = \frac{uc_d - dc_u}{(u - d)R_f} \quad (16)$$

- Hence, a portfolio of  $\Delta^*$  shares of stock and  $\$B^*$  of bonds produces the same cashflow as the call option.

## Binomial Option Pricing Example

- Therefore, the absence of arbitrage implies

$$c = \Delta^* S + B^* \quad (17)$$

where  $\Delta^*$  is the option's *hedge ratio* and  $B^*$  is the debt financing that are positive/negative (*negative/positive*) for calls (*puts*).

- *Example:* If  $S = \$50$ ,  $u = 2$ ,  $d = .5$ ,  $R_f = 1.25$ , and  $X = \$50$ , then  $uS = \$100$ ,  $dS = \$25$ ,  $c_u = \$50$ ,  $c_d = \$0$ .
- Therefore,

$$\Delta^* = \frac{50 - 0}{(2 - .5) 50} = \frac{2}{3}$$

# Binomial Option Pricing cont'd

$$B^* = \frac{0 - 25}{(2 - .5) 1.25} = -\frac{40}{3}$$

so that

$$c = \Delta^* S + B^* = \frac{2}{3} (50) - \frac{40}{3} = \frac{60}{3} = \$20$$

- This option pricing formula can be rewritten:

$$\begin{aligned} c &= \Delta^* S + B^* = \frac{c_u - c_d}{(u - d)} + \frac{uc_d - dc_u}{(u - d) R_f} & (18) \\ &= \frac{\left[ \frac{R_f - d}{u - d} \max[0, uS - X] + \frac{u - R_f}{u - d} \max[0, dS - X] \right]}{R_f} \end{aligned}$$

which *does not* depend on the stock's up/down probability,  $\pi$ .

## Binomial Option Pricing cont'd

- Since the stock's expected rate of return equals  $u\pi + d(1 - \pi) - 1$ , it need not be known or estimated to solve for the no-arbitrage value of the option,  $c$ .
- However, we do need to know  $u$  and  $d$ , the size of the stock's movements per period which determine its *volatility*.
- Note also that we can rewrite  $c$  as

$$c = \frac{1}{R_f} [\hat{\pi} c_u + (1 - \hat{\pi}) c_d] \quad (19)$$

where  $\hat{\pi} \equiv \frac{R_f - d}{u - d}$  is the *risk-neutral* probability of the up state.

- $\hat{\pi} = \pi$  if individuals are risk-neutral since

$$[u\pi + d(1 - \pi)] S = R_f S \quad (20)$$

which implies that

# Binomial Option Pricing cont'd

$$\pi = \frac{R_f - d}{u - d} = \hat{\pi} \quad (21)$$

so that  $\hat{\pi}$  does equal  $\pi$  under risk neutrality.

- Thus, (19) can be expressed as

$$c_t = \frac{1}{R_f} \hat{E} [c_{t+1}] \quad (22)$$

where  $\hat{E}[\cdot]$  denotes the expectation operator evaluated using the risk-neutral probabilities  $\hat{\pi}$  rather than the true, or physical, probabilities  $\pi$ .

# Multiperiod Binomial Option Pricing

- Next, consider the option's value with *two periods prior to maturity*. The stock price process is

$$\begin{array}{rcc}
 & & u^2 S \\
 & & \nearrow \searrow \\
 & uS & \\
 & \nearrow \searrow & \\
 S & & duS \\
 \nearrow \searrow & & \\
 & dS & \nearrow \searrow \\
 & & d^2 S
 \end{array} \quad (23)$$

so that the option price process is

# Multiperiod Binomial Option Pricing cont'd

$$\begin{array}{r}
 c_{uu} \equiv \max [0, u^2 S - X] \\
 \nearrow \\
 c_u \\
 \searrow \\
 c_{du} \equiv \max [0, duS - X] \\
 \nearrow \\
 c \\
 \searrow \\
 c_d \\
 \searrow \\
 c_{dd} \equiv \max [0, d^2 S - X]
 \end{array}
 \quad (24)$$

- We know how to solve one-period problems:

$$c_u = \frac{\hat{\pi} c_{uu} + (1 - \hat{\pi}) c_{du}}{R_f} \quad (25)$$

$$c_d = \frac{\hat{\pi} c_{du} + (1 - \hat{\pi}) c_{dd}}{R_f} \quad (26)$$

## Multiperiod Binomial Option Pricing cont'd

- With two periods to maturity, the next period cashflows of  $c_u$  and  $c_d$  are replicated by a portfolio of  $\Delta^* = \frac{c_u - c_d}{(u-d)S}$  shares of stock and  $B^* = \frac{uc_d - dc_u}{(u-d)R_f}$  of bonds. No arbitrage implies

$$c = \Delta^* S + B^* = \frac{1}{R_f} [\hat{\pi} c_u + (1 - \hat{\pi}) c_d] \quad (27)$$

which, as before says that  $c_t = \frac{1}{R_f} \hat{E} [c_{t+1}]$ .

- The market is complete over both the last period and second-to-last periods. Substituting in for  $c_u$  and  $c_d$ , we have

$$c = \frac{1}{R_f^2} \left[ \hat{\pi}^2 c_{uu} + 2\hat{\pi}(1 - \hat{\pi}) c_{ud} + (1 - \hat{\pi})^2 c_{dd} \right]$$

## Multiperiod Binomial Option Pricing cont'd

$$\begin{aligned}
 &= \frac{1}{R_f^2} \left[ \hat{\pi}^2 \max [0, u^2 S - X] + 2\hat{\pi} (1 - \hat{\pi}) \max [0, duS - X] \right] \\
 &\quad + \frac{1}{R_f^2} \left[ (1 - \hat{\pi})^2 \max [0, d^2 S - X] \right]
 \end{aligned}$$

which says  $c_t = \frac{1}{R_f^2} \hat{E} [c_{t+2}]$ . Note when a market is complete each period, it becomes *dynamically complete*. By appropriate trading in just two assets, payoffs in three states of nature can be replicated.

- Repeating this analysis for any period prior to maturity, we always obtain

$$c = \Delta^* S + B^* = \frac{1}{R_f} [\hat{\pi} c_u + (1 - \hat{\pi}) c_d] \quad (28)$$

## Multiperiod Binomial Option Pricing cont'd

- Repeated substitution for  $c_u, c_d, c_{uu}, c_{ud}, c_{dd}, c_{uuu}$ , and so on, we obtain the formula, with  $n$  periods prior to maturity:

$$c = \frac{1}{R_f^n} \left[ \sum_{j=0}^n \left( \frac{n!}{j!(n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \max [0, u^j d^{n-j} S - X] \right] \quad (29)$$

or  $c_t = \frac{1}{R_f^n} \hat{E} [c_{t+n}]$ . Define “ $a$ ” as the minimum number of upward jumps of  $S$  for it to exceed  $X$ .

- Then for all  $j < a$  (out of the money):

$$\max [0, u^j d^{n-j} S - X] = 0 \quad (30)$$

while for all  $j > a$  (in the money):

$$\max [0, u^j d^{n-j} S - X] = u^j d^{n-j} S - X \quad (31)$$

## Multiperiod Binomial Option Pricing cont'd

- Thus, the formula for  $c$  can be simplified:

$$c = \frac{1}{R_f^n} \left[ \sum_{j=a}^n \left( \frac{n!}{j!(n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} [u^j d^{n-j} S - X] \right] \quad (32)$$

- Breaking up (32) into two terms, we have

$$c = S \left[ \sum_{j=a}^n \left( \frac{n!}{j!(n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \left[ \frac{u^j d^{n-j}}{R_f^n} \right] \right] - XR_f^{-n} \left[ \sum_{j=a}^n \left( \frac{n!}{j!(n-j)!} \right) \hat{\pi}^j (1 - \hat{\pi})^{n-j} \right] \quad (33)$$

The terms in brackets are complementary binomial distribution functions, so that (33) can be written

## Multiperiod Binomial Option Pricing cont'd

$$c = S\phi[a; n, \hat{\pi}'] - XR_f^{-n}\phi[a; n, \hat{\pi}] \quad (34)$$

where  $\hat{\pi}' \equiv \left(\frac{u}{R_f}\right)\hat{\pi}$  and  $\phi[a; n, \hat{\pi}]$  is the probability that the sum of  $n$  random variables that equal 1 with probability  $\hat{\pi}$  and 0 with probability  $1 - \hat{\pi}$  is  $\geq a$ .

- Formula (34) can converge to the Black-Scholes option pricing formula as the period length goes to zero.
- Suppose each period is of length  $\Delta t$  and keep  $T = n\Delta t$  fixed but let  $\Delta t \rightarrow 0$  as  $n \rightarrow \infty$ .

## Multiperiod Binomial Option Pricing cont'd

- Next let  $u = e^{\sigma\sqrt{\Delta t}}$  and  $d = 1/u = e^{-\sigma\sqrt{\Delta t}}$ , which gives a stock return variance of  $\sigma^2$  per unit time.
- Then as the number of periods  $n \rightarrow \infty$ , but the length of each period  $\Delta t = \frac{T}{n} \rightarrow 0$ , the Central Limit Theorem implies that formula (34) converges to:

$$c = SN(z) - XR_f^{-T} N\left(z - \sigma\sqrt{T}\right) \quad (35)$$

where  $z \equiv \left[ \ln\left(\frac{S}{XR_f^{-T}}\right) + \frac{1}{2}\sigma^2 T \right] / \left(\sigma\sqrt{T}\right)$  and  $N(\cdot)$  is the cumulative standard normal distribution function.

# Summary

- Forward contract payoffs can be replicated using a static trading strategy.
- Option contract payoffs require a dynamic trading strategy.
- A dynamically complete market allows us to use risk-neutral valuation.
- Dynamically complete markets imply replication of payoffs in all future states, but we may need to execute many trades to do so.