EE 415 / EE 418 Game theory

Lecture 4 applications of Nash Equilibrium continued and Advanced theory

- **Final-offer Arbitration**
- **1.3 Advanced theory: mixed strategies and existence of equilibrium**

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Application: Final-offer arbitration

- Wage disputes are settled by (a) conventional arbitration : the arbitrator is free to impose any wage as the settlement; or (b) final-offer arbitration : the two sides make wage offers and then the arbitrator picks one of the offers as the settlement.
- Two players: firm and a union
- The dispute concerns wages.
- Set up: First, both simultaneously make offers, denoted by w_f and w_u.
- Second, the arbitrator chooses one of the two offers as the settlement

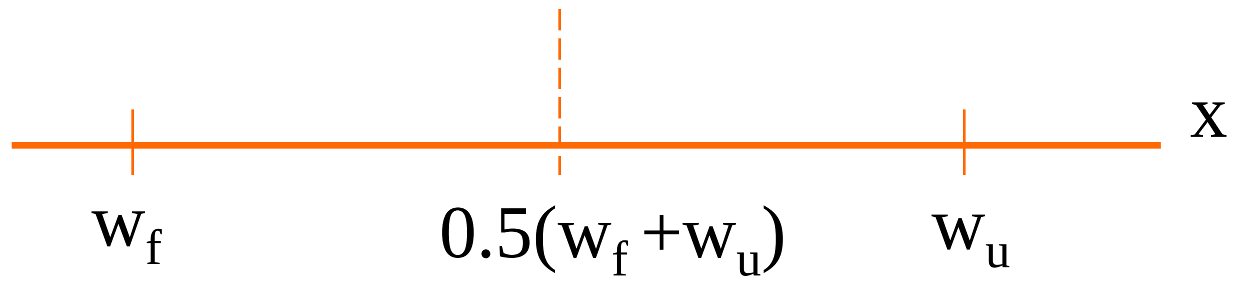
APPLICATION: FINAL-OFFER ARBITRATION

- Assume that the arbitrator has an ideal settlement she likes, denoted by x.
- After seeing both offers, she simple chooses the offer that is closer to x
- Given that $w_f < w_u$ (intuitive), choosing

-- w_f if $x < (w_f + w_u) / 2$ or

-- w_u if $x > (w_f + w_u) / 2$

-- flips a coin if $x = (w_f + w_u) / 2$



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- the arbitrator knows x but the parties do not.
- The parties believe that x ~ with a cumulative probability distribution F(x), and associated prob. density function f(x).
- Noting that the prob. that x is less than a value x* is denoted by F(x*).
- For any x*, $0 \leq F(x^*) \leq 1$. And if $x^{**} > x^*$, then $F(x^{**}) \geq F(x^*)$, so $f(x^*) \geq 0$ for every x*.
- $\int_a^b f(x)dx = prob(a \leq x \leq b)$

cdf pdf

Application: Final-offer arbitration

- Now, we define the parties believe that
- Prob $\{w_f \text{ chosen}\} = \text{Prob } \{x < (w_f + w_u) / 2\}$
- $= F((w_f + w_u) / 2)$
- And
- Prob $\{w_u \text{ chosen}\} = 1 - F((w_f + w_u) / 2)$
- Thus, the expected wage settlement is

  $w_f \text{ Prob } \{w_f \text{ chosen}\} + w_u \text{ Prob } \{w_u \text{ chosen}\}$

  $$= w_f F((w_f + w_u) / 2) + w_u [1 - F((w_f + w_u) / 2)]$$

- Next, we assume that the firm wants to minimize the expected wage settlement imposed by the arbitrator, and the union wants to maximize it.

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Application: Final-offer arbitration

- If the pair of offers (w_f^*, w_u^*) to be a NE, w_f^*must solve
- min w_f $F((w_f + w_u) / 2)$ + w_u [1 - $F((w_f + w_u) / 2)$]

- and w_u^*must solve
- max w_f $F((w_f + w_u) / 2)$ + w_u [1 - $F((w_f + w_u) / 2)$]

- Thus, NE must solve the FOC's for these optimization problem,
- $\left(w_f f(.).\frac{1}{2} + F(.) - w_u f(.)\frac{1}{2} \right) = 0$
- $\rightarrow (w_u - w_f)\frac{1}{2} f(.) = F(\,.\,)$ .....(1)

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- $and\left(w_f f(.).\frac{1}{2} + 1 - w_u f(.)\frac{1}{2} - F(.) \right) = 0$
- $\rightarrow (w_u - w_f)\frac{1}{2} f(.) = 1 - F(.)$ ……..(2)
- Since LHS of (1) and (2) are equal , thus RHS must be equal too.
- This implies that
- $F(.) = 1 - F(.) \quad \Longrightarrow \quad F(.) = 0.5$ …..(*)
- This means that the average of the offers must equal the median of the arbitrator's preferred settlement.
- Substitute (*) in any FOC will imply that
- $(w_u - w_f)\frac{1}{2} f(.) = 1 - 0.5 = 0.5$
- => $(w_u - w_f) = \frac{1}{f(.)}$. …………..(**)
- The gap b/w offers = 1 over value of pdf at the median of the arbitrator's preferred settlement.

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- $Now, suppose\ that\ the\ arbitrator's\ preferred$
- $settlement\ is\ normally\ distributed\ with\ mean\ m$
- $and\ variance\ \boldsymbol{\sigma}^2\quad, which$
- $f(x) = \frac{1}{\sqrt{2\boldsymbol{\pi\sigma}^2}} \exp\{-\frac{1}{2\boldsymbol{\sigma}^2}(x-m)^2$
- Under the normal, if $F(x) = 0.5, then\ x = mean.$
- $So\ we\ know\ that\ in$ (*) , $(w_f + w_u)/2 \ = m$
- and (**)=> $(w_u - w_f) = \frac{1}{f(m)}$=$\sqrt{2\boldsymbol{\pi\sigma}^2}$.
- So, the NE offers are
- $w_u = m + \sqrt{\frac{\boldsymbol{\pi\sigma}^2}{2}}\ and\ \ w_f = m - \sqrt{\frac{\boldsymbol{\pi\sigma}^2}{2}}$
- The parties' offers are centered around the expectation of the arbitrator's preferred settlement (m), and the gap increases with uncertainty about m, that is $\boldsymbol{\sigma}^2$.

- Intuition behind this equilibrium: each party faces a trade-off.
- Too aggressive (lower by firm vs higher by union) offer yields a better payoff if it is chosen as the settlement but it is less likely
- Same happens in a sealed-bid auction: lower bid yields a better payoff but reduces the chances of winning.
- What happen when uncertainty is larger. You can bid more aggressive far from the mean m.
- But if uncertainty is slim, neither party can afford to make an offer far from the mean because the arbitrator is very likely to prefer settlements close to m.
- Application: medical malpractice cases

ADVANCE THEORY: MIXED STRATEGIES

- Remind that in Matching penny game, there is no pure NE, in which each player would like to outguess the others.
- Similarly in games like poker (how often you want to bluff), battle (land or sea)
- In such a game, <u>solution</u> involves <u>uncertainty</u> about what the players will do.
- Notion of mixed strategy can be interpreted in terms of one player's uncertainty about what another player will do.
- A Mixed strategy for player i is a probability distribution over the strategies in Si
- In matching penny, a mixed strategy for player i is the prob distribution (q, 1-q) , where q is probability of playing Heads, and (1-q) is the probability of playing Tails.
- You can say that the mixed strategy (0,1) is simply the pure strategy Tails.

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ADVANCE THEORY: MIXED STRATEGIES

- Suppose player i has K pure strategies; $S_i = \{s_{i1}, \ldots, s_{iK}\}$
- **Definition** In G, a mixed strategy for player i is a probability distribution $p_i = (p_{i1},\ldots,p_{iK})$, where $0 \leq p_{iK} \leq 1$ for $k = 1, \ldots, K$ and $p_{i1} + \ldots + p_{iK} = 1$
- Remind that : if play x is strictly dominated, then you cannot form belief (about the strategies the other players will choose) that makes x be optimal.
- The converse is true if we allow for mixed strategies.
- So you can say that : if you cannot hold belief making x optimal, then there exist another strategy that strictly dominates x.

EVEN IF THE PURE STRATEGY IS NOT DOMINATED BY ANY OTHER PURE, IT MAY BE STRICTLY DOMINATED BY A MIXED STRATEGY

	L	R
T	3, ---	0 , --
M	0 , --	3 , --
B	1 , --	1 , --

B is not strictly dominated by T or M, but can be dominated by a mix of T and M with (0.5, 0.5), yielding expected payoff of 1.5 regardless of how player 2 plays, and this is more than 1.

Even if the pure strategy is not a best response to any other pure strategy, it may be a best response to a mixed strategy

	L	R
T	3, ---	0 , --
M	0 , --	3 , --
B	2 , --	2 , --

- B is not a best response for player 1 to either L or R by player 2,

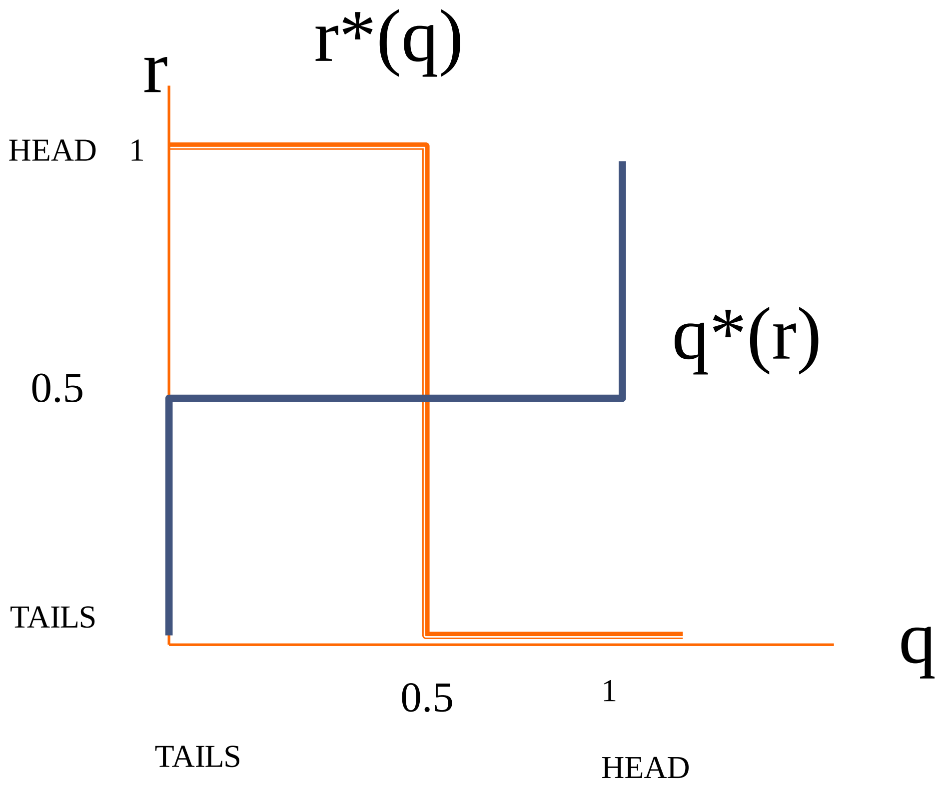
But B is the best response for player 1 to the mixed strategy (r, 1-r) by player 2 if $1/3 < r < 2/3$.

- This shows the role of mixed strategies in the belief that player 1 could hold.

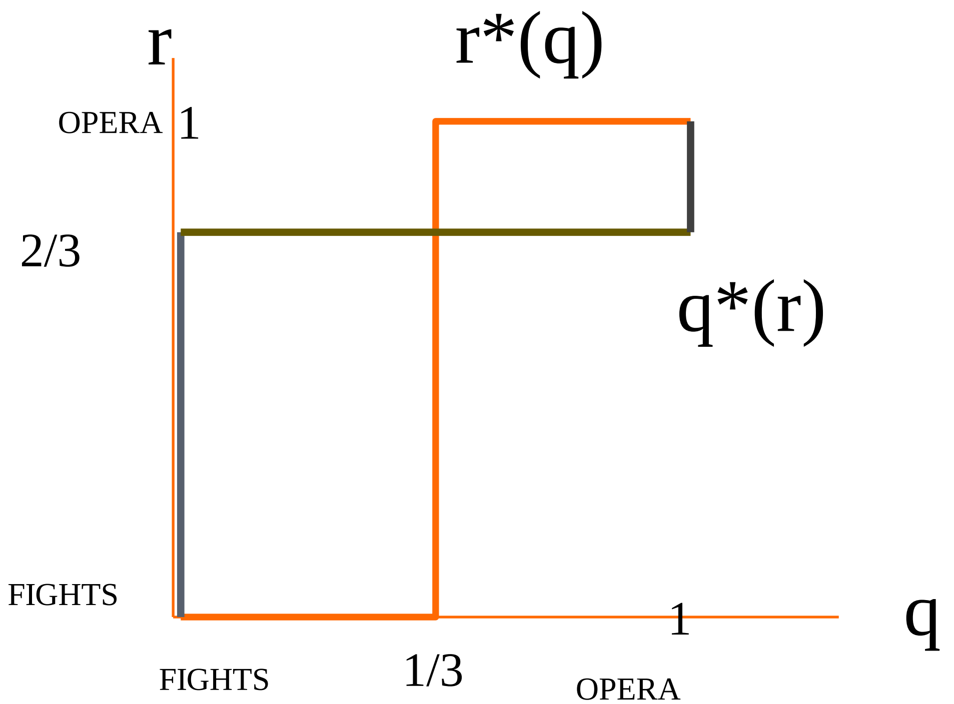
Existence of NE

- In any game, a NE appears as intersection of players' best response function (correspondence), even when there are more than two players, and even when some or all players have more than two pure strategies.
- Simple diagram in two-player games in which each has two strategies
- In text, we can draw the best response function for mixed strategy, say r(q) and q(r) in the matching penny game.
- As in Cournot model, the intersection of the best response correspondence yields the mixed-strategy Nash equilibrium.
- That is, if player 1 plays (0.5, 0.5), then (0.5, 0.5) is a best response for player 2.
- Noting that such a mixed-strategy NE does not rely on any player flipping coins, rolling dice or choosing a strategy at random.
- Rather, we interpret player 1's mixed strategy as a statement of player 1's uncertainty about player 2's choice of a pure strategy.

MATCHING PENNY GAME



BATTLE OF SEXES (3 INTERSECTIONS)



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Existence of NE

- Skip graphical argument that any 2 by 2 game has a NE (p.41-44)
- **Theorem (Nash 1950):** In the n-player normal-form game, if set of player is finite, and strategy set of each player S_i finite, then there exists at least one NE, possibly mixed strategies.
- Proof: using a fixed-point theorem, suppose f(x) is a continuous function with domain and range [0,1]. Then Brower's fixed point theorem guarantees that there exits at least one fixed point. That is, there exists at least one value of x^* in [0,1] such that $f(x^*) = x^*$.
- In a mixed NE, for each player, p^*_i must be the best response to other p^*_{-i}

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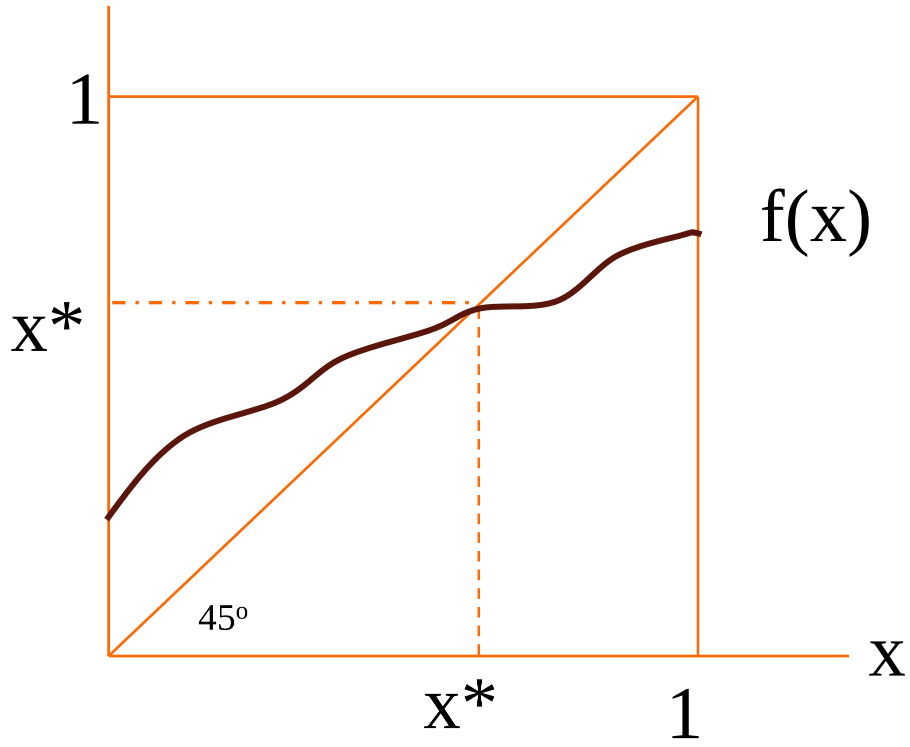
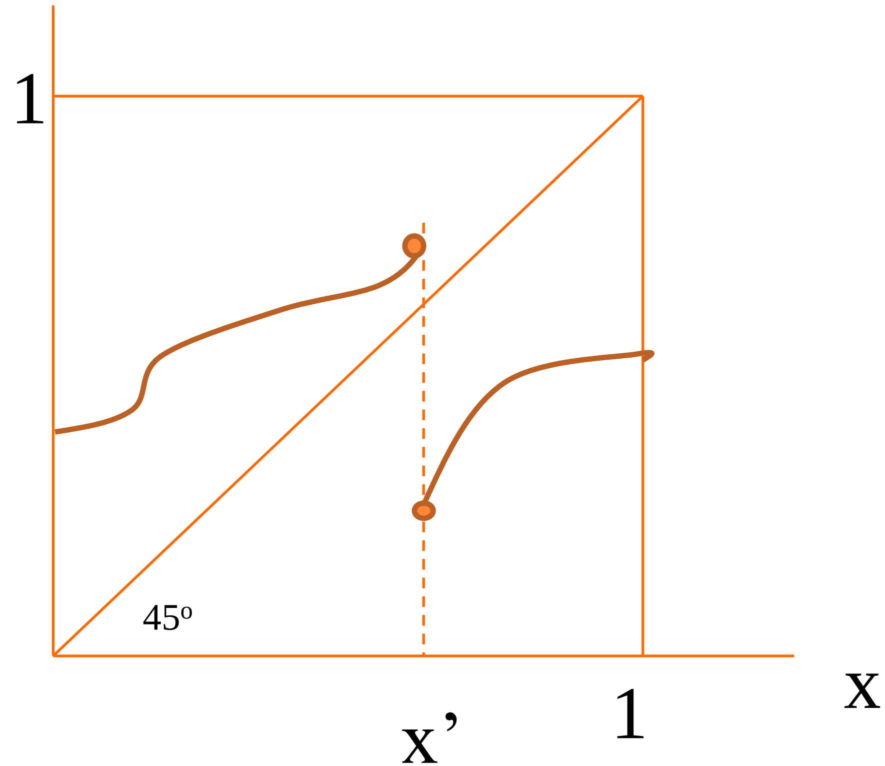
BROUWER'S FIXED POINT THEOREM



To prove NE's theorem, we need to show
(a) any fixed point of a certain correspondence is a NE; and
(b) this correspondence must have a fixed point.

KAKUTANI'S FIXED POINT THEOREM (ALLOWING FOR WELL-BEHAVED CORRESPONDENCE AS WELL AS FUNCTIONS)



At x', f is discontinuous and no fixed point.
For correspondence, f(x') would include not only the solid circle but also open circle and the entire interval in between, thus f(x) has a fixed point at x'.

FINAL NOTES

- So far, we look at application in which strategy set is not finite, but we can find NE anyway.
- This implies that the NE theorem is sufficient but not necessary conditions for an equilibrium to exist.
- Homework 1: section 1.1: Q1.2 , Q1.3
- Section 1.2:Q1.6, Q1.7
- Section 1.3: Q1.12, Q1.13

LET'S PLAY: CLASSROOM GAMES

- 2. The Ultimatum Game
- Two players
- Player A offers a split of a 100 Baht bill
- If B agrees, then the game is over.
- If B refuses, it is then B's turn to offer a split, but now the bill reduced to 80 Baht
- If A agrees, then the game is over. Both get paid the agreed split.
- If A refuses, the game is over and neither get anything.