Multiple Regression Analysis : Further Issues

1 Data scaling on OLS statistics

When we change the unit of measurment of a variable, the value of estimators would chance accordingly. For example

$$\widehat{bweght} = \widehat{\beta}_0 + \widehat{\beta}_1 cigs + \widehat{\beta}_2 fa\min c,$$

where
$bwght$ = child birth weight, in grams.
$cigs$ = number of cigarettes smoked by the mother while pregnant, per day.
$fa\min c$ = annual family income, in thousands of dollars.

- what if we use bweght in kg??

1kg = 1000g

$$\widehat{bwght}_{kg} = \frac{\widehat{bweght}}{1000} = \frac{\hat{\beta}_0}{1000} + \frac{\hat{\beta}_1}{1000} cigs + \frac{\hat{\beta}_2}{1000} famine$$

$$= \hat{\alpha}_0 + \hat{\alpha}_1 cigs + \hat{\alpha}_2 famine$$

$$\Rightarrow \hat{\alpha}_0 = \frac{\hat{\beta}_0}{1000} \quad \hat{\alpha}_1 = \frac{\hat{\beta}_1}{1000} \quad \hat{\alpha}_2 = \frac{\hat{\beta}_2}{1000}$$

- what if we use faminc in USD (instead of 1000 USD)

$$bwght_g = \hat{\beta}_0 + \hat{\beta}_1 cigs + \hat{\beta}_2 faminc_{USD}$$

↑ the value of this variable is going to be 1000 times larger than faminc

so, $\hat{\beta}_0 + \hat{\beta}_1 cigs + \hat{\theta}_2 faminc_{USD}$

$$\Rightarrow \hat{\theta}_2 = \frac{\hat{\beta}_2}{1000}$$, since β_2 = impact of 1000 USD in income

in other words $\hat{\theta}_2$ = impact of 1 USD in income

- what if we use bwght in kg & income in THB

$$bwght_{kg} = \frac{\hat{\beta}_0}{1000} + \frac{\hat{\beta}_1}{1000} cig + \frac{\hat{\beta}_2}{\frac{1000}{30,000}} faminc_{THB}$$

↑ this value going to be 30,000 times more than faminc

2 More on functional forms

- Logarithmic Functional Form

$$\log(y) = \beta_0 + \beta_1 \log(x_1) + \beta_2 x_2 + u$$

with the log y & log x format, the coefficient is going to be the elasticity! (x, elasticity of y)

$$\beta_1 = \frac{d\log(y)}{d\log(x)} = \frac{\frac{1}{y}dy}{\frac{1}{x_1}dx_1} = \frac{\frac{1}{y}\Delta y}{\frac{1}{x_1}\Delta x_1} = \frac{100 \times \frac{1}{y}\Delta y}{100 \times \frac{1}{x_1}\Delta x_1} = \frac{\%\Delta y}{\%\Delta x}$$

$\Delta y = y_1 - y_2$, $\Delta x = x_{11} - x_{12}$

$$\beta_2 = \frac{d\log(y)}{dx_2} = \frac{\frac{1}{y}dy}{dx_2} = \frac{\frac{1}{y}\Delta y}{\Delta x_2}$$

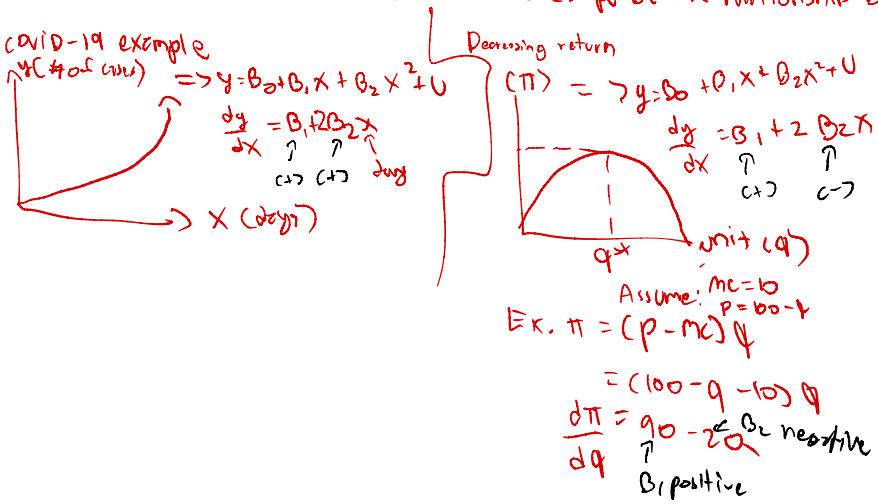
=> if we want the upper term to be % change then, $100\beta_2 = \frac{100\frac{1}{y}\Delta y}{\Delta x_2}$

$100\beta_2 = \frac{\%\Delta y}{\Delta x_2}$

so, $100\beta_2 = \%\Delta$ in y given that x_2 increase by 1 unit

- Models with Quadratics [squares]

-> capture inc/dec marginal effects (slope of the relationship b/w x&y is not constant.



Assume: mc = 10, P = 100 - q

Ex. $\pi = (p - mc)q$

$= (100 - q - 10)q$

$\frac{d\pi}{dq} = 90 - 2q$, β_2 negative

β_1 positive

Example : Effects of Pullution on Housing Prices

$$\log(price) = \beta_0 + \beta_1 \log(nox) + \beta_2 \log(dist) + \beta_3 rooms + \beta_4 room^2 + \beta_5 stratio + u$$

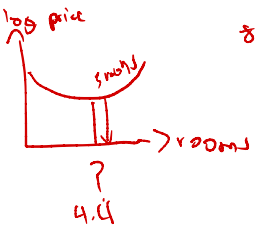
where
$price$ = housing price
nox = level of pollution
$dist$ = distance from downtown
$rooms$ = number of rooms
$stratio$ = average student per teacher ratio
The estimation result is given by

```
. regress lprice lnox dist rooms rooms_sq stratio
```

Source	SS	df	MS	Number of obs =	506
Model	51.4933152	5	10.298663	F(5, 500) =	155.62
Residual	33.0889098	500	.06617782	Prob > F =	0.0000
				R-squared =	0.6088
Total	84.582225	505	.167489554	Adj R-squared =	0.6049
				Root MSE =	.25725

lprice	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
lnox	-.9767545	.0995938	-9.81	0.000	-1.172429	-.7810806
dist	-.0321972	.0094013	-3.42	0.001	-.050668	-.0137264
rooms	-.5528032	.1612965	-3.43	0.001	-.8697056	-.2359007
rooms_sq	.0624697	.0124867	5.00	0.000	.0379368	.0870025
stratio	-.0486667	.0058131	-8.37	0.000	-.0600879	-.0372455
_cons	13.59154	.5650901	24.05	0.000	12.4813	14.70178

Consider the effect of "room"



What wold be the % change in price when the number of room increases from 5 to 6?

3 Models with Interaction Terms

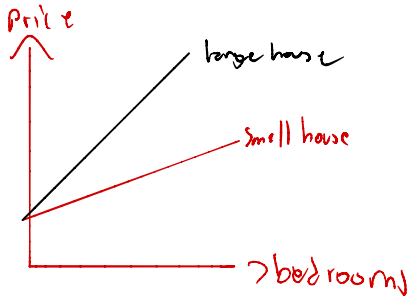
=> used when the impact of one variable depends on the value (level) of another variable

Consider

$$price = \beta_0 + \beta_1 sqrft + \beta_2 bdrms + \beta_3 sqrft \times bdrms + \beta_4 bthrms + u$$

x_1 x_2 x_3 $x_1 \cdot x_2$ x_2

where
- $price$ = housing price
- $sqrft$ = house size (square feet)
- $bdrms$ = number of bedrooms
- $bthrms$ = number of bathrooms



$\frac{\Delta price}{\Delta bdrms} = \beta_2 + \beta_3 sqrft$

=> if $\beta_2 > 0$ then, an additional bedroom would increase price more for a larger house

4 More on the Goodness-of-Fit and Selection of Regressors

- Adding more regressors ALWAYS improve fit → R^2 always increase

- But we lose the "degree of freedom"

→ 1 data point is sacrificed everytime we estimate a parameter.

→ using R^2 would not punish "having too many regressors"

→ we use adjusted R^2 or $\bar{R}^2$ when we want to punish adding too many regressors

$$R^2 = 1 - \frac{SSR}{SST} = 1 - \frac{SSR/n}{SST/n}$$

$$\text{adj } R^2 = 1 - \frac{SSR/(n-k-1)}{SST/(n-1)}$$

↳ if we have more k, $df = n-k-1 \downarrow$, $SSR/(n-k-1) \uparrow$, adj $R^2 \downarrow$

Using adjusted R-squared to choose between non-nested models (one model is not a subset of another).

Consider Model 1

$$\begin{aligned} \widehat{salary} &= \underset{(223.90)}{830.63} + \underset{(0.0089)}{0.0163}sales + \underset{(11.08)}{19.63}roe \\ n &= 209, \quad R^2 = 0.029, \quad \bar{R}^2 = 0.020 \end{aligned}$$

Consider Model 2

$$\begin{aligned} \widehat{\log(salary)} &= \underset{(0.29)}{4.36} + \underset{(0.033)}{0.2751}\log(sales) + \underset{(0.004)}{0.0179}roe \\ n &= 209, \quad R^2 = 0.282, \quad \bar{R}^2 = 0.275 \end{aligned}$$

↳ 27.5% of variation in y is explained so this model is better

Multiple Regression Analysis with Qualitative Information:

1 Outline

- Describing qualitative information
- Using a single dummy independent variable
- Using dummy variables for multiple categories
- Interactions involving dummy variables
- A binary dependent variable (Y variable): The linear probability model

2 Describing Qualitative Information

- "Female" and "Married" are qualitative variable.
- We arbitarily assign a dummy variable to decribe them.

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

$$married = \begin{cases} 1 & \text{if married} \\ 0 & \text{otherwise (of if single)} \end{cases}$$

TABLE 7.1

A Partial Listing of the Data in WAGE1.RAW

person	wage	educ	exper	female	married
1	3.10	11	2	1	0
2	3.24	12	22	1	1
3	3.00	11	2	0	0
4	6.00	8	44	0	1
5	5.30	12	7	0	1
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
525	11.56	16	5	0	1
526	3.50	14	5	1	0

3 Models with a single dummy independent variable

Consider

$$wage = \beta_0 + \delta_0 female + \beta_1 educ + u.$$

where

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

In this case, the δ_0 notation is used to highlight the interpretation of the parameters multiplying dummy variables. In other cases, we can use any notation that is the most convenient.

① $E(wage|female,educ) = E(\beta_0+\delta_0 female+\beta_1 educ+u \mid female,educ)$

$= \beta_0+\delta_0 female+\beta_1 educ+E(u|female,educ)$

$= \beta_0+\delta_0 female+\beta_1 educ$ $\quad$ $\overset{!!}{=}0$ [Ass MLR1-4 holds]

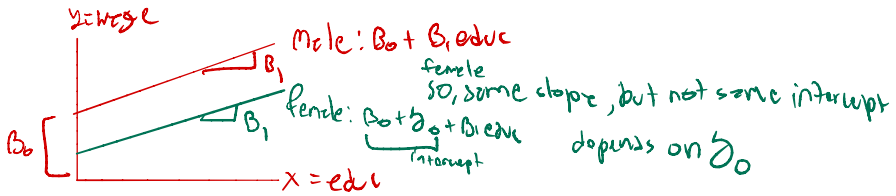
② Thus,

Female: $E(wage|female=1,educ) = \beta_0+\delta_0(1)+\beta_1 educ = \beta_0+\delta_0+\beta_1 educ$

Male: $E(wage|female=0,educ) = \beta_0+\delta_0(0)+\beta_1 educ = \beta_0+\beta_1 educ$

$\delta_0 = E(wage|female,educ) - E(wage|male,educ)$

* Given the same value of educ, δ_0 is the difference in the expected wage of females and males.



female
so, same slope, but not same intercept
depends on δ_0

∴ Assume $\delta_0 < 0$

→ By the way, female is going to give constant impact on wage regardless of level of educ

4 It is not possible to include all of the dummy alternatives in the same model (as long as there is intercept)

- If we include all alternatives of a dummy variable in the same model, we will face the "perfect collinearity" problem.

For example:

$$\begin{aligned} 1 &= female + male \\ female &= male + 1 \end{aligned}$$

or

If there are n categories, we omit "1" category to avoid multicollinearity

$$\begin{aligned} 1 &= winter + spring + summer + fall \\ winter &= 1 - spring - summer - fall \end{aligned}$$

winter = [1 if winter, 0 otherwise]
spring = [1 if spring, 0 otherwise]
etc.

id	W	Sp	Su	F	Xo
1	1	0	0	0	1
2	0	1	0	0	1
3					
4	1	0	0	0	1
	0	0	0	1	1

in this case male ↓

- At least one alternative has to be dropped. We treat the dropped alternative as the "BASE GROUP" or "BASELINE" or "BENCHMARK GROUP".

```
. regress lwage female male married educ exper
note: male omitted because of collinearity

      Source |       SS       df       MS              Number of obs =     526
-------------+------------------------------           F(  4,   521) =   75.27
       Model |  54.3265253     4  13.5816313            Prob > F      =  0.0000
    Residual |  94.0032262   521  .180428457           R-squared     =  0.3663
-------------+------------------------------           Adj R-squared =  0.3614
       Total |  148.329751   525  .28253286            Root MSE      =  .42477

------------------------------------------------------------------------------
       lwage |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      female |  -.3251146   .0377061    -8.62   0.000    -.3991892    -.25104
        male |           0  (omitted)
     married |   .1380145   .0411197     3.36   0.001     .0572338    .2187953
        educ |   .0872644   .0071554    12.20   0.000     .0732075    .1013213
       exper |   .0076213   .0015314     4.98   0.000     .0046129    .0106297
       _cons |   .4690918   .1040575     4.51   0.000      .264668    .6735156
------------------------------------------------------------------------------
```

Female workers are expected to get less wage compare to male workers

5 Using dummy variables for multiple categories

Case 1 We can use many dummy variables in the same model

Consider a model which includes 2 dummy variables– *female* and *married*.

$$\log(wage) = \beta_0 + \delta_0 female + \delta_1 married + \beta_1 educ + \beta_2 \exp er + \beta_3 \exp er^2 + \beta_4 tenure + \beta_5 tenure^2 + u.$$

```
regress lwage female married educ exper expersq tenure tenursq
```

Source	SS	df	MS		
Model	65.6482326	7	9.37831895	Number of obs	= 526
Residual	82.6815188	518	.159616832	F(7, 518)	= 58.76
				Prob > F	= 0.0000
Total	148.329751	525	.28253286	R-squared	= 0.4426
				Adj R-squared	= 0.4351
				Root MSE	= .39952

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.2901838	.0361121	-8.04	0.000	-.3611279	-.2192396
married	.0529219	.0407561	1.30	0.195	-.0271456	.1329894
educ	.0791547	.0068003	11.64	0.000	.0657952	.0925143
exper	.0269535	.0053258	5.06	0.000	.0164907	.0374163
expersq	-.0005399	.0001122	-4.81	0.000	-.0007603	-.0003196
tenure	.0312962	.0068482	4.57	0.000	.0178426	.0447499
tenursq	-.0005744	.0002347	-2.45	0.015	-.0010355	-.0001134
_cons	.4177837	.0988662	4.23	0.000	.2235557	.6120116

Comments:

1.) δ_0 measures the expected difference b/w female & male workers given the same marital status and other factors

$$\frac{d\log(wage)}{d\,female} = \frac{1}{wage}\frac{d\,wage}{d\,female} = -0.29$$

$$\frac{100\cdot\frac{1}{wage}d\,wage}{d\,female} = 100(-0.29)$$

$$\frac{\%\,\Delta wage}{d\,female} = -29.02\%$$

∴ female workers are expected to earn less than male workers by 29.02%.

2.) δ_1 measures impact of married

but since $|t| < 1.96$ or $p > 0.05$ we do not reject H_0 of No impact.

Consider a model which includes dummy variables for each gender/marital status combination– $marrmale$, $marrfem$ and $singfem$. [sing male used a the base case]

$$\log(wage) = \beta_0 + \delta_0 marrmale + \delta_1 marrfem + \delta_3 singfem + \beta_1 educ + \beta_2 exper + \beta_3 exper^2 + \beta_4 tenure + \beta_5 tenure^2 + u. \quad (8.1)$$

```
regress lwage marrmale marrfem singfem educ exper expersq tenure tenursq
```

Source	SS	df	MS
Model	68.3617623	8	8.54522029
Residual	79.9679891	517	.154676961
Total	148.329751	525	.28253286

Number of obs	= 526
F(8, 517)	= 55.25
Prob > F	= 0.0000
R-squared	= 0.4609
Adj R-squared	= 0.4525
Root MSE	= .39329

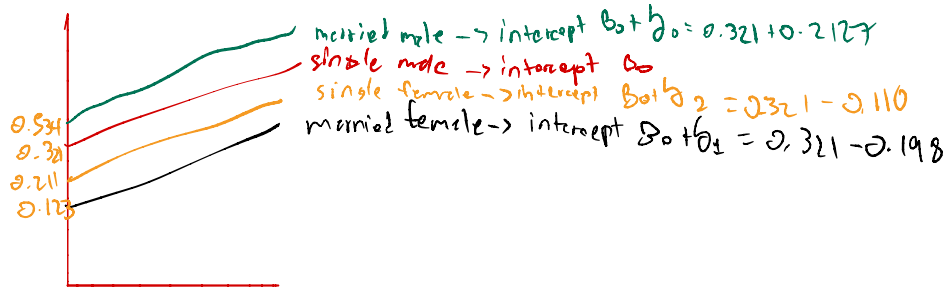
lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
marrmale	.2126757	.0553572	3.84	0.000	.103923	.3214284
marrfem	-.1982676	.0578355	-3.43	0.001	-.311889	-.0846462
singfem	-.1103502	.0557421	-1.98	0.048	-.219859	-.0008414
educ	.0789103	.0066945	11.79	0.000	.0657585	.092062
exper	.0268006	.0052428	5.11	0.000	.0165007	.0371005
expersq	-.0005352	.0001104	-4.85	0.000	-.0007522	-.0003183
tenure	.0290875	.006762	4.30	0.000	.0158031	.0423719
tenursq	-.0005331	.0002312	-2.31	0.022	-.0009874	-.0000789
_cons	.3213781	.100009	3.21	0.001	.1249041	.5178521

δ_0 δ_1 δ_2 $\hat{\beta}$

This regression is not same as previous one. It uses "single male" as the base group. (the previous one use male & single as 2 base group.)

Comments:

- δ_0 measures the expected diff in wage of married male as compared with single males, holding other constant.
- δ_1 | married female as compared with single males, holding other constant
- δ_2 -> same rationale



Case 2 We can use dummy variables to represent multiple categories of a variable Consider the relationship between law school rankings and starting salaries

$$\begin{aligned}\log(salary) &= \beta_0 + \delta_0 top10 + \delta_1 r11\_25 + \delta_3 r26\_40 + \delta_4 r41\_60 + \beta_1 LSAT \\ &\quad + \beta_2 GPA + \beta_3 \log(libvol) + \beta_4 \log(cost) + u.\end{aligned}$$

where $top10$, $r11\_25$, $r26\_40$, $r41\_60$ would be equal to 1 when the variable $rank$ falls into the appropriate range.

** Rank below 60 would be the base case.

In many cases the "range of values" serve as a better explanatory variable than the "value" itself. E.g. age may explain the model better if split into generation Young-as ages 16-29 etc.

```
. regress lsalary top10 r11_25 r26_40 r41_60 LSAT GPA llibvol lcost
```

Source	SS	df	MS		
Model	9.16538532	8	1.14567316	Number of obs	= 136
Residual	1.2109665	127	.009535169	F(8, 127)	= 120.15
				Prob > F	= 0.0000
Total	10.3763518	135	.076861865	R-squared	= 0.8833
				Adj R-squared	= 0.8759
				Root MSE	= .09765

lsalary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
top10	.5393428	.053542	10.07	0.000	.4333927	.6452928
r11_25	.4716199	.0390921	12.06	0.000	.3942637	.548976
r26_40	.2790977	.0346972	8.04	0.000	.2104383	.3477571
r41_60	.182382	.0283098	6.44	0.000	.126362	.238402
LSAT	.0060482	.0034919	1.73	0.086	-.0008616	.012958
GPA	.1305893	.0818678	1.60	0.113	-.0314122	.2925908
llibvol	.0725522	.0289213	2.51	0.013	.0153221	.1297824
lcost	.0249169	.0283224	0.88	0.381	-.031128	.0809619
_cons	8.363103	.4457314	18.76	0.000	7.481081	9.245125

the baseline is ranking 61th and worse

Comments:

1.) δ_0 measures difference in expected log(salary) of a law school graduate from a top 10 university compared to expected log(salary) of those who graduated from the school ranked 61th and worst

2.) $\delta_0 \rightarrow$ some notion