Chapter 9

Optimization with Equality Constraint with many choice variables

- Lagrange multiplier
- Conditions for optimization
- Maximize output level subject to cost constraint
- Minimize cost subject to output constraint
- ~~Minimize~~ maximize utility subject to budget constraint
- Minimize total expenditure under the level of utility

Transition of idea

Ch. 8 | Ch. 9

Unconstrained optimization | Constrained optimization

max / min $z = f(x_1, x_2)$
x_1, x_2

max / min $z = f(x_1, x_2)$
x_1, x_2

s.t. $h(x_1, x_2) = c$ ⇒ constraint

Lagrangian function

max / min $L = f(x_1, x_2) + \lambda [c - h(x_1, x_2)]$
x_1, x_2, λ

as if λ becomes another choice variable

FOC partial derivative w.r.t. every choice variables

SOC matrix that collects every possible 2nd order partial derivative of obj fn / Lagrangian fn

Hessian Matrix | Bordered Hessian matrix

What if choice variables cannot be chosen freely? For example, what if a firm cannot freely chooses quantities because of production quota? Firm will need to find constrained optimum value.

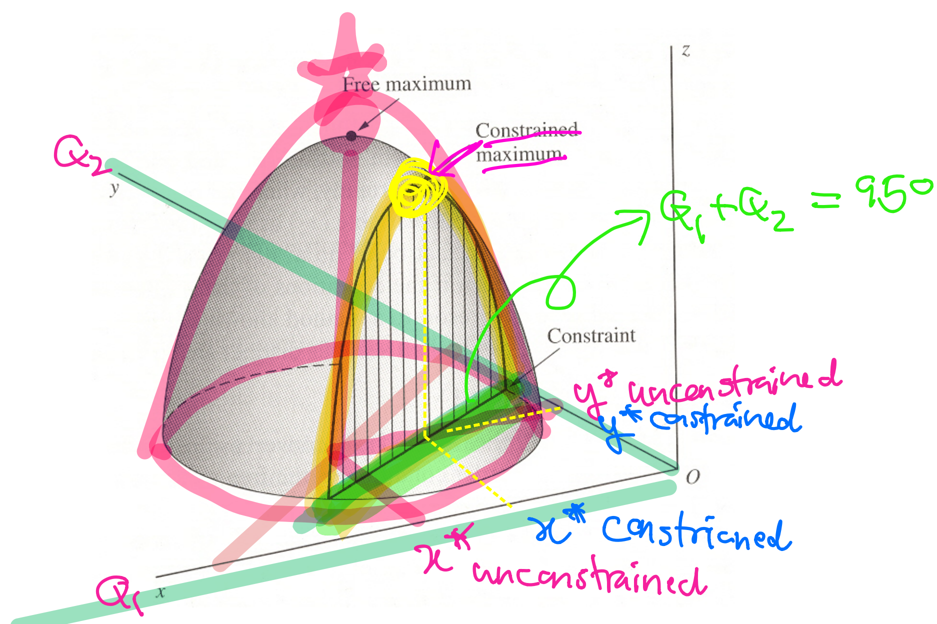
Suppose that a firm has two factories and is subjected to production quota of 950. The profit maximization problem becomes:

$$\max_{Q_1,Q_2} \pi(Q_1, Q_2)$$

$$\text{Subject to } Q_1 + Q_2 = 950$$

,where Q_1, Q_2 are quantity produced at factory 1 and 2, respectively.

Free Extremum vs. Constrained Extremum



Economics is about finding the optimal allocation of scarce resources. Hence, the main problems in economics can be studied as constrained optimization problem. The prototype of such problem is:

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\begin{aligned}
&\max imize \;\; f(x_1, x_2 ..., x_n) \\
&where\left(x_1, x_2, ..., x_n\right) \;\; must \;\; satisfy \\
&\qquad g_1\left(x_1, x_2, ..., x_n\right) \le b_1 \\
&\qquad g_2\left(x_1, x_2, ..., x_n\right) \le b_2 \\
&\qquad .... \\
&\qquad g_n\left(x_1, x_2, ..., x_n\right) \le b_n \\
&\qquad h_1\left(x_1, x_2, ..., x_n\right) = c_1 \\
&\qquad h_2\left(x_1, x_2, ..., x_n\right) = c_2 \\
&\qquad .... \\
&\qquad h_n\left(x_1, x_2, ..., x_n\right) = c_n
\end{aligned}
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Optimization with Equality Constraints :Two variables and One Equality Constraint

Let Mr. Musk have utility function $U = x_1x_2 + 2x_1$. If $P_1 = 4$ and $P_2 = 2$ Baht per piece, how many piece of good 1 and good 2 will Mr. Musk choose to consume to maximize his utility?

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Now suppose that Mr. Musk has the budget equal to 60 baht, how many piece of good 1 and good 2 will Mr. Musk choose to consume to maximize his utility?

"Lagrange Multiplier"

Let f and h be functions with two independent variables x_1, x_2

$$\max_{x_1, x_2} f(x_1, x_2)$$

$$\text{Subject to } h(x_1, x_2) = c$$

"Lagrangian Function"

$$L(x_1, x_2, \mu) = f(x_1, x_2) + \mu\left[c - h(x_1, x_2)\right]$$

We will call μ Lagrange multiplier.

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FONC:

$$\frac{\partial L}{\partial x_1} = f_1 - \mu h_1(x_1, x_2) = 0$$

$$\frac{\partial L}{\partial x_2} = f_2 - \mu h_2(x_1, x_2) = 0$$

$$\frac{\partial L}{\partial \mu} = c - h(x_1, x_2) = 0$$

Lagrangian function for utility maximization problem when Mr.Musk has 60 baht is:

$L =$

FONC:

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Homework: Write Lagrangian function, first order conditions, and critical values of the following problems:

(a.)

$$\begin{aligned} &Max\ U(x_1, x_2) = x_1 x_2 \\ &St. \quad h(x_1, x_2) = x_1 + 4x_2 = 16 \end{aligned}$$

(b.)

$$\begin{aligned} &Max\ z = xy \\ &St. \quad x + y = 6 \end{aligned}$$

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Optimization with Equality Constraints : n variables and One Equality Constraint

$$f(x_1, x_2, \dots, x_n)$$

$$st.\ h(x_1, x_2, \dots, x_n) = c$$

Lagrangian function is

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꧁ Second-Order Sufficient Condition ꧂

Consider objective function: $f(x_1, x_2, ..., x_n)$

$$\begin{aligned} St. \quad & h^1(x_1, x_2, ..., x_n) = c_1 \\ & h^2(x_1, x_2, ..., x_n) = c_2 \\ & .... \\ & h^k(x_1, x_2, ..., x_n) = c_k \end{aligned}$$

The Lagrangian

$$L(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) + \mu_1\left[c_1 - h^1(x_1, x_2, ..., x_n)\right] + \mu_2\left[c_2 - h^2(x_1, x_2, ..., x_n)\right] + ... \mu_k\left[c_k - h^k(x_1, x_2, ..., x_n)\right]$$

<u>First Order Necessary Condition: FONC</u>

$L_{x_1} = \dfrac{\partial L}{\partial x_1} = 0$ ....

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$L_{x_n} = \dfrac{\partial L}{\partial x_n} = 0$ ....

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$L_{\mu_1} = \dfrac{\partial L}{\partial \mu_1} = 0$ ....

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$L_{\mu_k} = \dfrac{\partial L}{\partial \mu_k} = 0$ ....

<u>Second Order Sufficient Condition: SOSC</u>

"Bordered Hessian Matrix"……………………..

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We can divide Bordered Hessian Matrix into 4 parts:

Note: We evaluate Bordered Hessian at the critical values.

$$\bar{H}=\begin{bmatrix} L_{\mu_1\mu_1} & L_{\mu_1\mu_2} & \cdots & L_{\mu_1\mu_k} & L_{\mu_1x_1} & L_{\mu_1x_2} & \cdots & L_{\mu_1x_n} \\ L_{\mu_2\mu_1} & L_{\mu_2\mu_2} & \cdots & L_{\mu_2\mu_k} & L_{\mu_2x_1} & L_{\mu_1x_2} & \cdots & L_{\mu_1x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{\mu_k\mu_1} & L_{\mu_k\mu_2} & \cdots & L_{\mu_k\mu_k} & L_{\mu_kx_1} & L_{\mu_kx_2} & \cdots & L_{\mu_kx_n} \\ L_{x_1\mu_1} & L_{x_1\mu_2} & \cdots & L_{x_1\mu_k} & L_{11} & L_{12} & \cdots & L_{1n} \\ L_{x_2\mu_1} & L_{x_2\mu_2} & \cdots & L_{x_2\mu_k} & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{x_n\mu_1} & L_{x_n\mu_2} & \cdots & L_{x_n\mu_k} & L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}_{(n+k)\times(n+k)}$$

$$\bar{H}=\begin{bmatrix} 0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\ 0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\ -h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ -h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}_{(n+k)\times(n+k)}$$

We can have submatrix H_1 with border, $\bar{H}_1$, and so on. Try to get the submatrix from $\bar{H}$.

$$\bar{H}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\ 0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\ -h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ -h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}_{(n+k)\times(n+k)}$$

$$\bar{H}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -h_1^1 & -h_2^1 & \cdots & -h_n^1 \\ 0 & 0 & \cdots & 0 & -h_1^2 & -h_2^2 & \cdots & -h_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -h_1^k & -h_2^k & \cdots & -h_n^k \\ -h_1^1 & -h_1^2 & \cdots & -h_1^k & L_{11} & L_{12} & \cdots & L_{1n} \\ -h_2^1 & -h_2^2 & \cdots & -h_2^k & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_n^1 & -h_n^2 & \cdots & -h_n^k & L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}_{(n+k)\times(n+k)}$$

Question:
How many submatrices do we need to consider?

Answer:
$n - k$ matrices, i.e. the last n-k leading principle minors of matrix $\bar{H}$.

n is the number of choice variables and k is the number of constraints.

We need to focus since $\bar{H}_{k+1}$.

We need to check:

$$\bar{H}_{k+1},\ \bar{H}_{k+2}, \ldots \bar{H}_n = \bar{H}$$

Example 1: the case of one constraint with two choice variables:

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix}$$

Number of choice variables (n) ………..

Number of constraint (k) ………..

So we need to check the last n-k = …………………………..Matrix, which is:

$$\bar{H}_{k+1} = \bar{H}_{1+1} = \bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} = \bar{H}$$

Example 2: 3 choice variables and 1 constraint

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ -h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix}$$

Number of choice variables (n) ………..

Number of constraint (k) ………..

So we need to check the last n-k = …………………………..Matrix, beginning at $\bar{H}_{k+1}$ =

$$\bar{H}_2 = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{bmatrix} \qquad \bar{H}_3 = \bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ -h_{x_3} & L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix}$$

We need to observe pattern of the determinants of leading principal submatrices as the following.

The objective function will be at the local max if the last (n-k) leading principal minors have *alternate signs with* $|\bar{H}_{k+1}|$ has the same sign as $(-1)^{k+1}$.

The objective function will be at the local min if the last (n-k) leading principal minors have the *same sign* as $(-1)^{k}$ for every bordered Hessian that we need to consider.

ᏒSOC of 2 choice variables + 1 equality constraint ℘

$$|\bar{H}_2| = \begin{vmatrix} 0 & -h_{x_1} & -h_{x_2} \\ -h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ -h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{vmatrix}$$

If $|\bar{H}_2| > 0$ ➔

If $|\bar{H}_2| < 0$ ➔

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FOC, SOC ของ n choice variables + 1 equality constraint

$$f(x_1, x_2, ..., x_n)$$

$$St. \quad g(x_1, x_2, ..., x_n) = c$$

$$L = f(x_1, x_2, ..., x_n) + \lambda[c - g(x_1, x_2, ..., x_n)]$$

FOC:

$$L_1 = L_2 = L_3 = ... = L_n = L_\lambda = 0$$

SOC:

MAX

$|\bar{H}_2| > 0$; $|\bar{H}_3| < 0$; $|\bar{H}_4| > 0$;..........

MIN

$|\bar{H}_2|$; $|\bar{H}_3|$; $|\bar{H}_4|$;...; $|\bar{H}_n| < 0$

$$\bar{H} = \begin{bmatrix} 0 & -h_{x_1} & -h_{x_2} & -h_{x_3} & \cdots & -h_{x_n} \\ -h_{x_1} & L_{11} & L_{12} & L_{13} & \cdots & L_{1n} \\ -h_{x_2} & L_{21} & L_{22} & L_{23} & \cdots & L_{2n} \\ -h_{x_3} & L_{31} & L_{32} & L_{33} & \cdots & L_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_{x_n} & L_{n1} & L_{n2} & L_{n3} & \cdots & L_{nn} \end{bmatrix}$$

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Maximizing output level subject to cost constraint

Let the production function be $Q = q(L, K)$

The total cost is:

$$TC = wL + rK = C$$

Question: How will firm choose L and K to maximize output level, given that the total cost must be C baht.

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<u>Step 1:</u> Lagrangian Function

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<u>Step 2:</u> First-Order Condition (FONC):

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Economic Interpretation

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<u>Step 3:</u> Second-Order Sufficient Condition

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} z_{\lambda\lambda} = & z_{\lambda L} = & z_{\lambda K} = \\ z_{L\lambda} = & z_{LL} = & z_{LK} = \\ z_{K\lambda} = & z_{KL} = & z_{KK} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n = _______ k = ________

We need to check the last ______ leading principle Minor, which is___________

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Minimizing cost given a level of output:

Least Cost Combination of Inputs and Conditional Input Demand

Assume that a firm needs to produce output equal to Q_0 units from production function $Q = F(L, K)$, and the firm would like to choose the level of capital and labor that minimize the cost.

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<u>**Step 1:**</u> **Lagrangian Function**

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<u>**Step 2:**</u> **First-Order Condition (FONC):**

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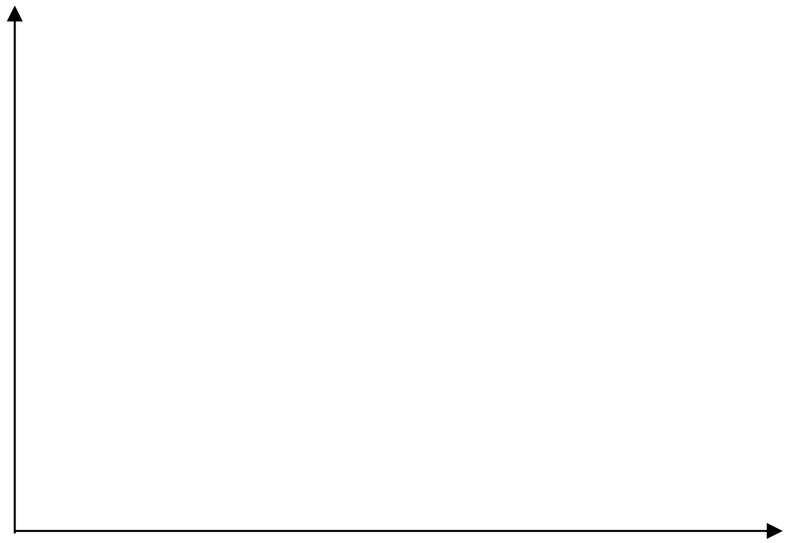
Economic Interpretation



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Step 3 **Second-Order Sufficient Condition**

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} z_{\lambda\lambda} = & z_{\lambda L} = & z_{\lambda K} = \\ z_{L\lambda} = & z_{LL} = & z_{LK} = \\ z_{K\lambda} = & z_{KL} = & z_{KK} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n =_________ k = _________

We need to check the last________ leading principle Minor นั่นคือ

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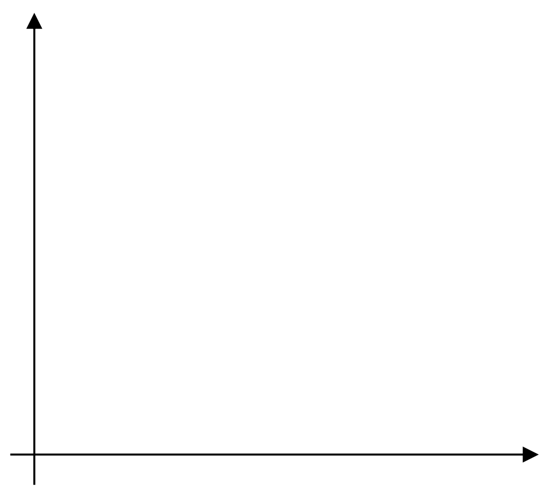
The Expansion Path

Expansion Path is the least cost combination between L* and K* given many levels of given outputs.



If Isoquants are strictly convex, expansion path can be found from the first order conditions.

If a firm has Cobb-Douglas production function $Q = AL^{\alpha}K^{\beta}$,

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In this case, the optimal ratio of capital to labor is constant and is equal to______

Conditional factor demand vs. Factor demand

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Cost minimization problem vs. Profit maximization problem

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Duality

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Utility maximization subject to budget constraint

$$U = U(x, y) \quad (U_x, U_y > 0)$$

$$\text{St.} \quad xP_x + yP_y = B$$

<u>Step 1</u>: Lagrangian Function

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<u>Step 2</u>: First-Order Condition (FONC):

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Economic Interpretation

Interpretation 1 : Marginal utility per one baht paid

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Interpretation 2: Indifference Curve

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Meaning of Lagrange multiplier

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Step 3 **Second-Order Sufficient Condition**

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} = & L_{\lambda x} = & L_{\lambda y} = \\ L_{x\lambda} = & L_{xx} = \frac{\partial^2 U}{\partial x^2} = & L_{xy} = \frac{\partial^2 U}{\partial x \partial y} = \\ L_{y\lambda} = & L_{yx} = \frac{\partial^2 U}{\partial y \partial x} = & L_{yy} = \frac{\partial^2 U}{\partial y^2} = \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n =_________ k = __________

Check the last ________ leading principle Minor :

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Suppose that Mr. Musk's utility function takes the form:

$$U(x, y) = 10x^{\frac{1}{2}}y^{\frac{1}{2}}$$

Mr. Musk has m baht. Price of Good x is P_x, and price of good y is P_y. What will be the level of x and y that maximize the utility?

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Lagrangian function:

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FOC:

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SOC:

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Minimize Total Expenditure Under the Level of Utility

Suppose now that a consumer would like to have utility U_0. Her utility is $U = U(x, y)$.

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<u>Step 1</u> Lagrangian Function

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<u>Step 2</u> First-Order Condition (FONC):

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<u>Step 3</u>: Second-Order Sufficient Condition

3.1 Construct Bordered Hessian

$$\bar{H} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

3.2 Check the last n-k leading principle Minor

n = ______ k = ______

Check the last________leading principle Minor:

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Income expansion path (Income consumption curve: ICC)

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Duality: Utility Maximization Problem vs. Expenditure Minimization Problem

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Other optimization problems:

Multiproduct firm with technological constraint

Quantities of two products are $x\&y$.

How many of $x\&y$ should the firm choose to produce to minimize the cost of production subjecting to technological constraint?

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Multiplant firm who knows exact total market demand

Let factory 1 produce Q_1 with cost $C_1 = C_1(Q_1)$; factory 2 produce Q_2 with cost $C_2 = C_2(Q_2)$; and let market demand be Q_0.

To minimize cost, how many should the firm produce at each factory given that it wants to meet the market demand?

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