Practice Problems - Final Exam

1. Let a and b be real numbers with $0 < b < a < 1$. Determine whether each of the following inequalities is true or not. Explain your answer.

   (a) $\frac{(b+1)^3}{|a|} < \frac{(a+1)^3}{b}$

   (b) $\frac{b}{a} < \frac{a^3-1}{b^3-1}$

2. Let x and y be real numbers with $y > x > 1$. Show that

$$y(y-2) > |x-1|^2 - 1.$$

3. Find the solution set for each of following inequalities.

   (a)
$$\frac{x^2-1}{x} < \frac{x+1}{2x}$$

   (b)
$$2 \leq \left|\frac{x-1}{x}\right| \leq 7$$

   (c)
$$\frac{x^2+|x|+1}{x^7+x^5+x^2+1} \leq 0$$

   (d)
$$\frac{|x+3|-2}{5} + \frac{1}{|x-1|+1} \leq 1$$

   (e)
$$\frac{|x-1|-x^2-1}{5-|x+3|} \geq 0$$

4. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define relations
   $f : X \to Y$ by $f = \{(1, a), (2, a), (3, c)\}$,
   $g : X \to Y$ by $g = \{(1, a), (3, c)\}$, and
   $h : X \to Y$ by $h = \{(1, a), (2, a), (3, b), (3, c)\}$.

   (a) Draw the arrow diagrams of f, g, and h.

   (b) Show that f is a function, but g and h are not functions.

   (c) Find the domain of f, co-domain of f,and range of f.

   (d) What is the inverse image of a for the function f?

   (e) What is $f(3)$?

5. Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For all $(x, y) \in A \times B$,

$$x \; R \; y \Leftrightarrow \; |x| = |y|.$$

and

$$x \; S \; y \;\; \Leftrightarrow \;\; x - y \text{ is even.}$$

State explicitly the sets $A \times B$, R, S, $R \cup S$, and $R \cap S$.

6. Let $A = \{1, 2, 3\}$ and $\mathbb{Z}$ be the set of all integers. Let $\mathcal{P}(A)$ be the set of all subsets of the set A, and

$$X = \{x \in \mathcal{P}(A) | \; x \cap \{1\} \neq \emptyset\}.$$

Define a relation r from X to $\mathbb{Z}$ as

$$r = \{(x, y) \in X \times \mathbb{Z} | \; y = \text{ the number of elements in } x\}.$$

(a) List all elements in X.

(b) Draw an arrow diagram of r .

(c) Is r a function? If so, find the domain, co-domain, and range of r.

7. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (x + 1, 2y - 3) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

8. Define functions $f_1 : [0, 2) \to \mathbb{R}$ as
$$f_1(x) = x^2$$
and define $f_2 : [2, \infty) \to \mathbb{R}$ as
$$f_2(x) = 3x - 2.$$
Let $F : [0, \infty) \to [0, \infty)$ be a function defined by using f_1 and f_2:
$F(x) = f_1(x)$, for $x \in [0, 2)$, and $F(x) = f_2(x)$, for $x \in [2, \infty)$. That is,
$$F(x) = \begin{cases} x^2, & x \in [0, 2) \\ 3x - 2, & x \in [2, \infty). \end{cases}$$
   (a) Find the domain, co-domain, and range for each of the functions f_1, f_2, and F.
   (b) Construct the composite functions $f_1 \circ f_2$, $f_2 \circ f_1$, and $f_1 \circ F$ (if possible). Determine the domains and ranges for these composite functions.
   (c) Are f_1 and f_2 injective? Explain.
   (d) Is the function F bijective? If so, find the **inverse function** of F.

9. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.
   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}$, $g(x) = x^3 + 1$.
   (b) $f(x) = x^5$, $g(x) = x^{1/5}$.

10. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.
   (a) Compute $f \circ f$ and determine its domain.
   (b) Determine whether f is bijective. If so, find the inverse function f^{-1} and $f \circ f^{-1}$.

11. Consider the relation f defined from X to Y, $X, Y \subseteq \mathbb{R}$,
$$f = \left\{ (x, y) \in X \times Y \,\middle|\, x^3 + yx^2 - y = 0 \right\}.$$
Note: It might be useful to write y as a function of x, i.e. $y = f(x)$, before answering some questions.
   (a) Find the domain of f.
   (b) Find x-intercepts and y-intercepts (if any).
   (c) Determine the symmetry of f.
   (d) Find the horizontal and vertical asymptotes for f (if any).
   (e) Show that $y = -x$ is a slant asymptote of f.
   (f) Find the critical number of f. Determine the intervals on which f is increasing and decreasing. Determine the extrema (maximum and minimum) of f.
   (g) Determine the intervals on which f is concave up and concave down. Find the points of inflection (if any).

(h) Sketch the curve of f.

12. Find all solution(s) of the following systems of linear equations by using **Gaussian elimination method**.

(a)
$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 6 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 16 \\ 2x_1 & + & 3x_2 & + & 5x_3 & + & 6x_4 & = & 25 \\ x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & = & 11 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 3x & + & 2y & + & z & = & 1 \\ 7x & + & 2y & - & 3z & = & 1 \end{array}$$

13. Consider the system of 3 equations and 3 unknowns x, y, z:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ x & + & 3y & + & 4z & = & 3 \\ x & + & 4y & + & kz & = & m, \end{array}$$

where m and k are some constants.

(a) Suppose $k = 5$. Find the value of m that makes the above system of equations have infinitely many solutions.

(b) Suppose $m = 4$. Find the value of m that makes the above system of equations have no solution.

14. Write out the appropriate form of the partial fraction decomposition of the given expressions Do not evaluate the coefficients.

(a)
$$\frac{x+1}{2x^2 - 11x + 15}$$

(b)
$$\frac{1}{x^4(x-1)(x^2+4)^3(x^2+x+1)^2}$$

15. Determine the partial fraction decomposition of

$$\frac{x^4 + 7x^3 + 8x^2 - 3}{x^3 + 6x^2}.$$