Assignment 5

1. Let $X = \{2, 3, 4\}$ and $Y = \{1, 5\}$. Define relations
   $f : X \to Y$ by $f = \{(x, y) \in X \times Y | x > y\}$,
   $g : X \to Y$ by $g = \{(x, y) \in X \times Y | x = 2y\}$, and
   $h : X \to Y$ by $h = \{(x, y) \in X \times Y | y = x^2\}$.

   (a) List all the elements of the Cartesian product $X \times Y$.

   (b) List all the elements of the relations f, g, and h.

   (c) Determine which of the relations f, g, h is a function from X to Y. Explain your answer.

   (d) What is the inverse image of 1 under f? What is the inverse image of 5 under f?

2. Let $A = \{1, 2, 3\}$ and let $\mathcal{P}(A)$ be the set of all subsets of the set A. Define a relation r and s as

$$r = \{(x, y) \in A \times \mathcal{P}(A) \mid x = \text{ the number of elements in } y\},$$

$$s = \{(u, v) \in \mathcal{P}(A) \times A \mid (v, u) \in r\}.$$

   (a) Draw arrow diagrams of r and s.

   (b) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.

   (c) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.

3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (y, x - 2) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

   (a) Is H one-to-one? Prove or give a counterexample.

   (b) Is H onto? Prove or give a counterexample.

   (c) Is H bijective? If so, find H^{-1}, the inverse function of H.

4. Define functions f and g as follows:
   $f = \{(1, 10), (3, 30), (5, 50)\}$ and
   $g = \{(10, k), (20, \ell), (30, m), (40, n), (50, t)\}$.

   (a) Determine the domain and range for each of functions f and g.

   (b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.

   (c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.

5. Define

$$g(x) = \frac{1}{\sqrt{x+1}} + 1 \qquad \text{and} \qquad F(x) = \begin{cases} x^2 - 1, & x \in [-3, 1) \\ 2 - x, & x \in [1, 4]. \end{cases}$$

   (a) Find the domain and range for each of the functions g and F.

   (b) Construct the composite functions $F \circ g$, and $g \circ F$ (if possible). Determine the domains for these composite functions.

   (c) Is the function F bijective? If so, find the **inverse function** of F. Justify your answer.

Optional Problems

1. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$

   (b) $f(x) = x^5, \quad g(x) = x^{1/5}.$

2. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.

   (a) Show that f is bijective.

   (b) Determine the inverse function f^{-1}.

   (c) Compute $f \circ f$ and $f \circ f^{-1}$.

3. Define $f : \mathbb{Z} \to \mathbb{Z}$ by the rule $f(n) = 2 - 3n$, for all integers n.

   (i) Is f one-to-one? Prove or give a counterexample.

   (ii) Is f onto? Prove or give a counterexample.

4. Define $G : \mathbb{R} \to \mathbb{R}$ by the rule $G(x) = 2 - 3x$ for all real numbers x. Is G onto? Prove or give a counterexample.

5. Define $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by $f(x) = \frac{x+1}{x}$, for all real numbers $x \neq 0$. Determine whether or not f is one-to-one and justify your answer.

6. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{x^2+1}$, for all real numbers x. Determine whether or not f is one-to-one and justify your answer.

7. Define $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows: $G(x, y) = (2y, -x)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

   a. Is G one-to-one? Prove or give a counterexample.

   b. Is G onto? Prove or give a counterexample.