EE320 (1/2013)

INTRODUCTORY MATHEMATICAL ECONOMICS

INTEGRATION AND ITS APPLICATION

Topics

- Terminology in Integration
- Indefinite Integration
- Basic Rules of Integration
- Definite Integration
- Improper Integrals

What is Integration?

- *Integration* is the inverse of differentiation.
- Formally, an integral is a function F(X) whose derivative is f(x):

$$F'(x) = f(x)$$

- This function F(x) can then be called an 'anti-derivative' of f(x).

Example:

- $f(x) = nx^{n-1}$ ➔ F(x) =
- $f(x) = \frac{1}{x}$ ➔ F(x) =

- The process of anti-differentiation is called integration.
- That is, to integrate a function f(x) is to find F(X) such that

$$F'(x) = f(x)$$

Indefinite Integrals

- If we integrate a function f(x) where values of x are not given, we have to *integrate without a limit* (i.e. *to find indefinite integral*).
- A symbol for integrating a function f(x) is:

$$\int f(x)dx = F(x) + C \Leftrightarrow F'(x) = f(x)$$

Where $\int$ is called the *integral sign*.

f(x) is called the *integrand*.

C is called the *constant of integration*.

dx indicates the variable involved in the integration.

- Note: A function does not have a unique integral.

Example: If $f(x) = 3x^2$, then F(x) =

Basic Rules of Integration

Rule I. Power rule: $\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$

Rule II. Exponential rule: $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

Rule III. Logarithmic rule: $\int \frac{1}{x} dx = \ln|x| + c$

Rule IV: Integral of a sum

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Rule V: Integral of a multiple

$$\int a f(x) dx = a \int f(x) dx$$

Rules of Operations

Rule IIa:

$$\int a^{bx}dx = \frac{1}{b\ln a}a^{bx} + c$$

Rule IIb:

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

Rule IIIa:

$$\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c, [f(x) \neq 0]$$

Examples

Find integrals of the following functions:

- $\int \frac{1}{x^3} dx$
- $\int \sqrt{x\sqrt{x\sqrt{x}}}\, dx$
- $\int (3x^4 + 5x^2 - 2) dx$
- $\int (e^{3x} - e^{2x} + e^x) dx$
- $\int \frac{(y-2)^2}{\sqrt{y}} dx$

Rules Involving Substitution

Rule VI: The Substitution Rule

$$\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c$$

Rule VII: Integration by Parts**

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Let u = f(x) and v = g(x). Then,

$$\int vdu = uv - \int udv$$

Examples: Integration involving substitution

- Examples: Find integrals of

a. $\int xe^{x}dx$

b. $\int \frac{1}{x}\ln(x)dx$

Initial-Value Theorem

- From $\int f(x)dx = F(x) + C$

If we have an initial condition, we can determine the value of C.

Example 1: Find F(x) if F'(x) = ½ - 2x and F(0) = ½.

Example 2: Find F(x) if F'(x) = $x(1-x^2)$ and F(1) = 5/12.

Application 1: Derivation of TR from MR

- $TR = \int MR(Q)dQ$
- Example: Let MR = 10 – 2Q , what is the TR function?

Application 2: Derivation of TC from MC

- $TC = \int MC(Q)dQ$
- Example: Let $MC = C'(Q) = 2e^{0.2Q}$.

  If the fixed cost is C_F = 90, what is the TC function?

Application 3: Derivation of Profit Function from MR-MC

- π' = MP = MR – MC. ➔ $\pi = \int \pi'(Q)dQ$
- Example: Let MR = 50 – 2Q and MC = 10 + Q. Find total profit when Q = 10. Assume that there is no fixed cost.

Application 4: Derivation of Utility Function from MU

- $U(x) = \int MU(x)dx$
- Example: Let $MU(x) = \dfrac{5}{3\sqrt{x}}$. Find the utility function.

Application 5: Derivation of Consumption/Saving Functions from Marginal Propensity Function

- Suppose the marginal propensity to save (MPS) function is:

  $S'(Y) = 0.3 - 0.1Y^{1/2}$, and S = 0 when Y = \$81.

  Find the saving and consumption functions.

Definite Integrals

- $\int_a^b f(x)dx$ is a "definite" integral of f(x) from a to b (a < b), where
  - a = the lower limit of integration
  - b = the upper limit of integration

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where F(x) = an arbitrary indefinite integral of f(x).

- Example: Find integral of
  - $\int_1^5 3x^2 dx$
  - $\int_0^1 \alpha e^{\beta\tau} d\tau$

Area and Definite Integrals

- The area under the graph of a continuous and nonnegative function f(x) over the interval [a, b] is $\int_a^b f(x)dx$.
- [Graph]

Properties of Definite Integrals (I)

Property I: The interchange of the limits of integration changes of the sign of the definite integral:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property II: A definite integral has a value of zero when the two limits of integrations are identical:

$$\int_a^a f(x)dx = 0$$

Property III: A definite integral can be expressed as a sum of a finite number of definite sub-integrals as follows:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, (a < b < c < d)$$

Properties of Definite Integrals (II)

Property IV:

$$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

Property V:

$$\int_a^b \alpha f(x)dx = \alpha\int_a^b f(x)dx$$

Property VI:

$$\int_a^b \left[f(x)+g(x)\right]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Property VII: (Integration by part)

$$\int_{x=a}^{x=b} vdu = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} udv$$

Examples: Definite Integrals

Find the integrals of

- $\int_{0}^{5}(x+x^{2})dx$
- $\int_{2}^{4}x^{2}(\frac{1}{3}x^{3}+1)dx$
- $\int_{-2}^{2}(e^{x}-e^{-x})dx$
- $\int_{e}^{6}(\frac{1}{x}+\frac{1}{1+x})dx$
- $\int_{-2}^{3}\left|x+1\right|dx$

Application 1: Capital Formation and Investment functions

- Definitions:
  - *K(t)* = capital stock at time t
  - *dK/dt* = the rate of capital formation
  - *I(t)* = the rate of net investment flow at time t
- Relationship between capital stock and net investment:

$$\frac{dK}{dt} = \dot{K} \equiv I(t) \quad \Rightarrow \quad K(t) = \int I(t)dt = \int \frac{dK}{dt}dt = \int dK$$

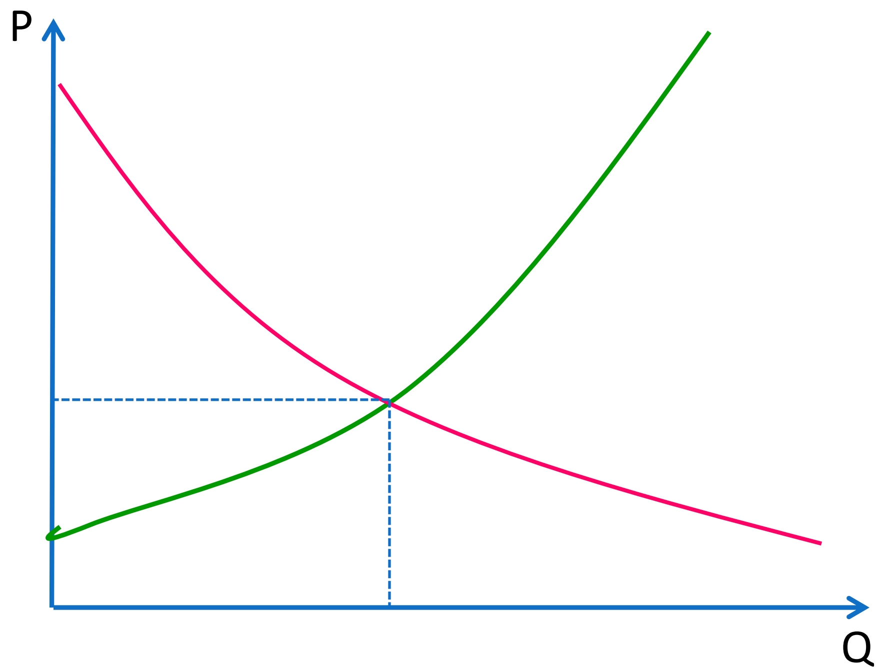
- Gross investment: $I_g(t) = I(t) + \delta K(t)$
- Capital formation during a time interval [a, b]:

$$\int_a^b I(t)dt = K(t)\Big]_a^b = K(b) - K(a)$$

Capital Formation and Investment functions (Cont'd)

- Example: Suppose the net investment flow is $I(t) = 3t^{1/2}$ and the initial capital stock at time t = 0 is $K(0) = 25$.
  1. What is the time path of capital K?
  2. What is the capital formation during the time interval [1, 4]?

Application 2: Consumer & Producer Surpluses



$$CS = \int_0^{Q^*} [D(Q) - P^*]dQ$$

$$PS = \int_0^{Q^*} [P^* - S(Q)]dQ$$

Example: Consumer Surplus & Producer Surplus

- Given a supply function $S(P) = -\frac{1}{2} + \frac{1}{2}P$ and a demand function $D(P) = \frac{25}{2} - \frac{1}{2}P$. Calculate: 1) producer and consumer surplus and 2) total welfare.

Welfare Effects of Price Change

- From the previous example, if the demand changes so that $D(P) = \frac{30}{2} - \frac{1}{2}P$, what happens to consumer and producer surplus?

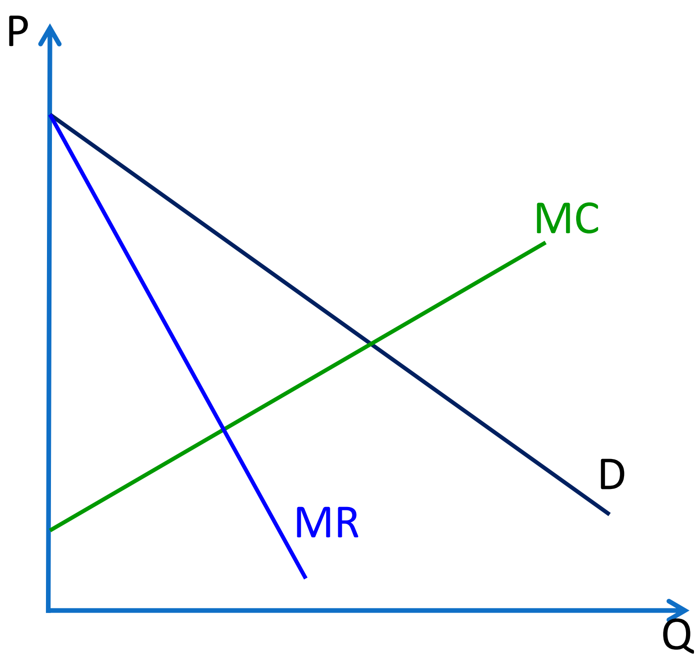
Welfare Effects of Tax Distortion

- From the previous example, if the government imposes a $4 per unit tax on the producer, calculate the total welfare loss.

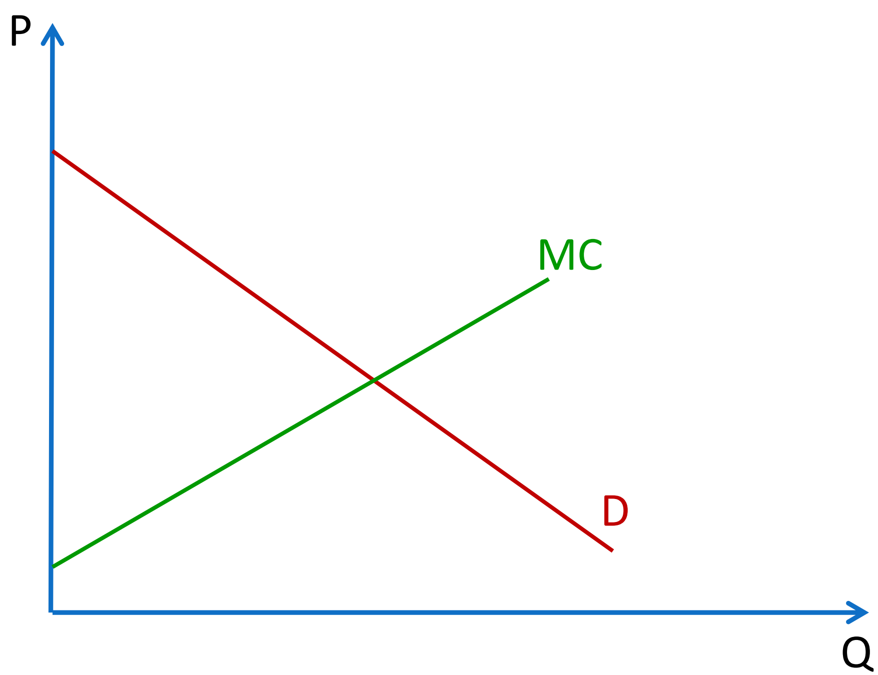
Application 3: First-Degree (or Perfect) Price Discrimination

- Perfect price discrimination occurs when the monopolist can charge the *maximum price for each unit of output sold*.

- (Regular) Monopolist



- Perfect price discriminating monopolist



Example: Perfect Price Discrimination (1)

- Suppose that a monopolist faces a demand function P = 24-Q and MC = 4 + 3Q. Find the consumer surplus at the profit-maximizing quantity. (No price discrimination)

Example: Perfect Price Discrimination (2)

- If the monopolist can practice perfect price discrimination, find the total revenue that maximizes the profit.

Improper Integrals (I)

- Case 1: When we have definite integrals of the form

$$\int_{a}^{\infty} f(x)dx \quad \text{and} \quad \int_{-\infty}^{b} f(x)dx$$

with one limit of integration being infinite, we refer to them as *improper integrals*. (because F(∞) and F(-∞) does not exist.)

➢ The improper integral can be defined to be the limit of another (proper) integral as follows:

$$\int_{a}^{\infty} f(x)dx \equiv \lim_{b\to\infty} \int_{a}^{b} f(x)dx$$

and

$$\int_{-\infty}^{b} f(x)dx \equiv \lim_{a\to-\infty} \int_{a}^{b} f(x)dx$$

- ➢If the *limit exists*, the improper integral is said to be *convergent* and the limiting process gives the value of the integral.
- ➢If the *limit does not exist*, the improper integral is said to be *divergent* and does not have any meaning.

Improper Integrals (II)

Examples: Evaluate the following integrals.

- $\int_{1}^{\infty} \frac{1}{x^2} dx$

- $\int_{1}^{\infty} \frac{1}{x} dx$

Improper Integrals (III)

- Case 2: Even with finite limit of integration, an integral can still be improper if the integrand becomes infinite in the interval of the integration [a, b].

Example: $\int_0^1 \frac{1}{x} dx =$

$$\int_0^9 \frac{1}{x^{1/2}} dx$$

- If *f(x)* is a continuous function for every $x \in R$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

Example: $\int_{-1}^{1} \frac{1}{x^3} dx =$