Note Title 8/28/2014

Topic #2 Differential Equations

DE is an equation.

(1) The unknown is a f^{n}., not a number

i.e, a f^{n} of time 't'

't' is an indepent variable

(2) The equation includes one or more derivatives of the f^{n}

Ex

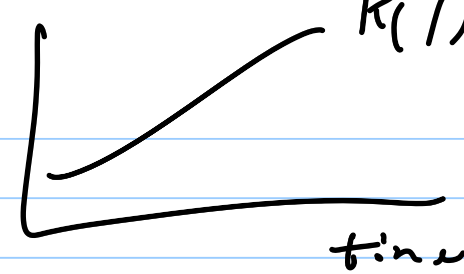
$$\frac{dy(t)}{dt} = a.\, y(t)$$

a, b ⇒ constants

$$\frac{dy(t)}{dt} + a.\, y(t) = b$$

$$\frac{dk(t)}{dt} = s.\, f(k(t)) - \lambda.k(t)$$

$$\frac{dy(t)}{dt} + 2.t.y(t) = 4.t$$



$$\frac{dy(t)}{dt} = y'(t) = \dot{y}(t)$$

In general, $\quad y'(t) = F(y(t), t)$

where $y(t)$ is the unknown $f^{\underline{n}}$

$\therefore$ The sol$^{\underline{n}}$ $\Rightarrow$ $y(t)$

The time path of $y(t)$

Note

$$y + 2 = 5$$

2.1 First-order Differential Equations

$\rightarrow$ only the first derivative $\Rightarrow \frac{dy(t)}{dt}, \left[\frac{dy(t)}{dt}\right]^2$

→ Focus on "linear" form

no non-linear term such as $y(t) \cdot y'(t)$

The general form

$$\boxed{y'(t) + u(t) \cdot y(t) = w(t)} \quad \text{—— (1)}$$

$\left.\begin{matrix} u(t) \\ w(t) \end{matrix}\right\}$ are two functions of time t

and could be constants → $u(t) = a$

$w(t) = b$

2.1.1 Constant coefficient and constant term

From (1), $u(t) = a$ and $w(t) = b$

$$y'(t) + a\,y(t) = b$$

Solve for the sol^{n} of $y(t)$

① Homogeneous Case

$$u = a \quad \text{and} \quad w = b = 0$$

$$\frac{dy(t)}{dt} + a.y(t) = 0 \qquad \text{——— (2)}$$

$$\frac{dy(t)}{dt} = -a.y(t)$$

$$\frac{1}{y}\frac{dy}{dt} = -a$$

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int (-a)\,dt$$

$$\int \frac{1}{y}\,dy = -at + c_2$$

$$\ln|y| + c_1 = -at + c_2$$

$$\ln|y| = -at + C \quad ; \quad C = C_2 - C_1$$

$$e^{\ln|y|} = e^{-at+C}$$

A General Sol^n

$$\boxed{y(t) = A \cdot e^{-at}} \quad \text{--- (3)}$$

where $A = e^C$

A Definite Sol^n

At $t=0$ $\quad y(0) = A$

$$\boxed{y(t) = y(0)\, e^{-at}} \quad \text{--- (3)'}$$

NOTE: Sol^ns are a f^n, not a number

$y(t)$ is a f^n of t

③ The Nonhomogeneous Case

$u = a$ and $w = b$; $b \neq 0$

$$\frac{dy(t)}{dt} + a\,y(t) = b \qquad -(4)$$

(4) ⇒ The complete equation $(b \neq 0)$

(2) ⇒ The reduced equation $(b = 0)$

$$\boxed{y(t) = y_c + y_p}$$

⇒ y_c = the complementary f^n

y_c is the sol^n when $b = 0$

$$\therefore \quad y_c = Ae^{-at}$$

⇒ y_p = the particular integral

To find y_p, we set y_p to be a constant $\Rightarrow y_p = k$

$\Rightarrow y'(t) = 0$

(4) becomes $0 + a y_p = b$

$$\boxed{y_p = \frac{b}{a}}$$ as long as $a \neq 0$

$$\boxed{k = \frac{b}{a}}$$

A General Sol^{n}

$$y(t) = y_c + y_p$$

$$\boxed{y(t) = A e^{-at} + \frac{b}{a}} \quad \text{—— (5)}$$

$a \neq 0$

A Definite Sol^{n}

At $t=0$ $\quad y(0) = A + \frac{b}{a}$

$$A = y(0) - \frac{b}{a}$$

$$\boxed{y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}} \quad \text{--- (6)}$$

$a \neq 0$

Verification of the sol^n - (6)

Diff (6) w.r.t. t $\quad y'(t) = (-a)\left[y(0) - \frac{b}{a}\right]e^{-at}$ — $(6)'$

sub $(6)'$ and $(6) \rightarrow$ LHS of (4)

$$y'(t) + a\,y(t) = b$$

$$(-a)\left[y(0) - \frac{b}{a}\right]e^{-at} + a\left\{\left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}\right\} = b$$

$$-a\,\cancel{[\quad]}\,e^{-at} + a\,\cancel{[\quad]}\,e^{-at} + a\left(\frac{b}{a}\right) = b$$

$$b = b$$

Also from (6)

At $t = 0$

$$y(0) = \left[y(0) - \frac{b}{a}\right]e^{-a \cdot 0} + \frac{b}{a}$$

$$= y(0) - \frac{b}{a} + \frac{b}{a}$$

$$\boxed{y(0) = y(0)}$$

<u>Ex</u>

$$\frac{dy}{dt} + 2y - 8 = 0 \Rightarrow a = 2 \quad b = 8$$

$$\boxed{\text{not } b = -8}$$

$$y(t) = y_c + y_p$$

$$y_c = \left[y(0) - \frac{b}{a}\right] e^{-at}$$

$$y_c = \left[y(0) - \frac{8}{2}\right] e^{-2t}$$

$$= (y(0) - 4)\, e^{-2t}$$

$$y_p = \frac{b}{a} = 4$$

$$y(t) = (y(0) - 4)\, e^{-2t} + 4$$

Equ (4) again

$$y'(t) + a\, y(t) = b \qquad — (4)$$

$b = 0 \Rightarrow$ Homogeneous case $\Rightarrow y_c$

$a = 0 \Rightarrow \longrightarrow (4)$ becomes

$y(t) = 0 \quad y = 0 \;\; \forall t \Rightarrow$ No dynamics

$y'(t) = 0 \Rightarrow y_p$?

$a = 0 \Rightarrow$ (4) becomes $y'(t) = b$

$$\boxed{y(t) = b.t + c}$$

; $c =$ constant

$$y(t) = y_c + y_p$$

from $y_c = Ae^{-rt}$ At $t=0$ $Ae^{0} = A$

and $y_p = \frac{b}{a} = k$ fails to work

since we set $y'(t) = 0$ to find y_p, this implies $y_p = k$ a constant

we try $y_p = k.t$

so $y(t) = y_c + y_p$

$$\boxed{y(t) = C + kt}$$ where $k = b$

General Sol^n $\quad y(t) = A + bt \quad a = 0 \quad$ — (7)

Definite Sol^n $\quad y(t) = y(0) + bt \quad a = 0 \quad$ — (8)

2.1.2 Dynamics of Market Price

① Theoretical Framework

$$\left.\begin{aligned} Q_d &= \alpha - \beta P \quad (\alpha, \beta > 0) \\ Q_s &= -\gamma + \delta P \quad (\gamma, \delta > 0) \end{aligned}\right\} \quad -(9)$$

$$P^* = \frac{\alpha + \gamma}{\beta + \delta} \quad -(10)$$

$P(0) = P^*$ → the mkt is in the equilibrium

$\Rightarrow$ no dynamic analysis might not be needed

If $P(0) \neq P^*$ $\qquad$ $P(0) > P^*$

$P(t) \rightarrow P^*$ $\qquad$ $P(0) < P^*$