Revision of Differentiation

1. Definition of Derivatives

Notation: $f'(x)$, y', $\frac{dy}{dx}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Constant Function Rule

$$\frac{d}{dx}(c) = 0$$

The derivative of any constant function is zero.

3. Power Rule

$$\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$$

The derivative of x to the power n is n times x to the power n minus 1.

4. General Power Rule (This is a special case of the **chain rule**.)

$$\frac{d}{dx}\left(u^n\right) = n\,u^{n-1}\frac{du}{dx}$$

5. Constant Multiply Property

$$\frac{d}{dx}\left[c\ f(x)\right] = c\ f'(x)$$

The derivative of a constant times a differentiable function is the constant times the derivative of the function.

6. Sum and Difference Property

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = f'(x) \pm g'(x)$$

The derivative of the sum/difference of two differentiable functions is the sum/difference of the derivatives.

7. Product Rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

8. Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

The derivative of the quotient of two functions is the bottom function times the derivative of the top function minus the top function times the derivative of the bottom function, all over the bottom function squared.

9. Chain Rule (Very useful differentiation rule!)
If $y = f(u)$ and $u = g(x)$ define the composite function $y(x) = f\left[g(x)\right]$, then the **chain rule** implies that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

10. Derivatives of Exponential Function

$$\frac{d}{dx}\left(e^n\right) = e^n \text{ where } e = 2.718\ 281\ 828\ 459\ \ldots$$

$$\frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$$

$$\frac{d}{dx}\left(a^u\right) = a^u (\ln a) \frac{du}{dx} \quad \text{(General form)}$$

11. Derivatives of Logarithmic Function
The logarithmic function $y = \log_b x$ is equivalent to the exponential function $x = b^y$.
If the base $b = e$, then the function is the natural logarithmic function and commonly written as $y = \ln x$.
If the base $b = 10$, then the function is the logarithmic function with the base 10 and commonly written as $y = \log x$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u} \cdot \frac{du}{dx}$$

12. Implicit Differentiation (Useful differentiation technique)
If the function is not given in the usual form $y = f(x)$ e.g. an equation likes $F(x, y) = 0$ and it is difficult to arrange the function into the usual form, then we can solve using **implicit differentiation** to find an equation in x, y, y'. The procedures are :-

12.1 Differentiate both sides of the equation with respect to *x*.

12.2 Collect all terms involving $\frac{dy}{dx}$ on one side of the equation, and collect all other terms on the other side.

12.3 Factor $\frac{dy}{dx}$ from the side involving the term $\frac{dy}{dx}$ terms.

12.4 Solve for $\frac{dy}{dx}$.

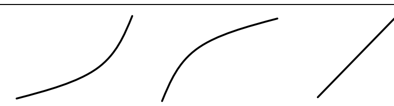
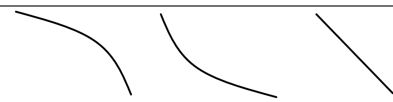
13. Logarithmic Differentiation (Useful differentiation technique)
This is a useful technique to simplify the differentiation of $y = f(x)$ when $f(x)$ involves products, quotients, or powers. The procedures are :-

13.1 Take the natural logarithm of both sides. This results in

$$\ln y = \ln[f(x)]$$

13.2 Simplify $\ln[f(x)]$ by using properties of logarithms.
13.3 Differentiate both sides with respect to x.
13.4 Solve for $\frac{dy}{dx}$.
13.5 Express the answer in terms of x only. This requires substituting $f(x)$ for y.

14. Increasing and Decreasing Functions
For the interval (a,b),

$f'(x)$	$f(x)$	Graph of f	Examples
+	Increase ↑	Rising ↑	
-	Decreasing ↓	Falling ↓	

15. Critical values
The values of x_c in the domain of *f* where $f'(x_c) = 0$ or where $f'(x_c)$ does not exist are called the **critical values** of *f*.

We call $f(x_c)$ a **local maximum** if there exists an interval (m,n) containing x_c such that

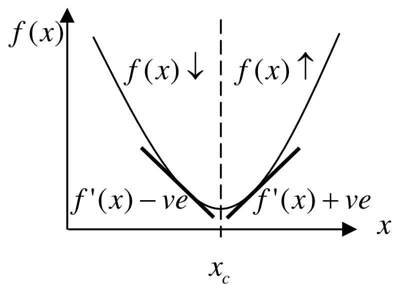
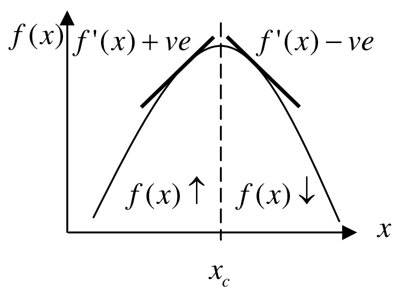
$$f(x) \le f(x_c) \quad \text{for all } x \text{ in } (m,n)$$

and a **local minimum** if there exists an interval (m,n) containing x_c such that

$$f(x) \ge f(x_c) \quad \text{for all } x \text{ in } (m,n).$$

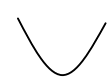
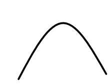
The quantity $f(x_c)$ is called a **local extremum** if it is either a local maximum or a local minimum. A point on a graph where a local extremum occurs is also called **a turning point**.

16. First-Derivative Test for Local Extrema
Let x_c be a critical value of *f* [$f(x_c)$ defined and either $f'(x_c) = 0$ or $f'(x_c)$ not defined]. Construct a sign chart for $f'(x)$ close to and on either side of x_c.

Sign Chart	$f(x_c)$
$f'(x)$ - - - + + + x_c $f(x)$ Decreasing Increasing 	If $f'(x)$ changes from negative to positive at x_c as x is increasing, then $f(x_c)$ is a local minimum.
$f'(x)$ + + + - - - x_c $f(x)$ Increasing Decreasing 	If $f'(x)$ changes from positive to negative at x_c as x is increasing, then $f(x_c)$ is a local maximum.
$f'(x)$ - - - - - - x_c $f(x)$ Decreasing Decreasing	$f(x_c)$ is not a local extremum. If $f'(x)$ does not change sign at x_c, then $f(x_c)$ is neither a local maximum nor a local minimum.
$f'(x)$ + + + + + + x_c $f(x)$ Increasing Increasing	$f(x_c)$ is not a local extremum. If $f'(x)$ does not change sign at x_c, then $f(x_c)$ is neither a local maximum nor a local minimum.

17. Second Derivative Test

Let x_c be a critical value of f [$f(x_c)$ defined and either $f'(x_c)=0$ or $f'(x_c)$ not defined].

$f'(x_c)$	$f''(x_c)$	Graph of f	$f(x_c)$	**Shape**
0	+	Concave up	Relative minimum	
0	-	Concave down	Relative maximum	
0	**0**	???	Neither min. nor max.	

18. Inflection Points

If $f''(x)=0$ or $f''(x)$ does not exist but $f'(x)$ does not need to be zero, $f(x_c)$ is an **inflection point**. An **inflection point** is a point on the graph of the function where the concavity changes (from upward to downward or from downward to upward). For the concavity to change at a point, $f''(x)$ must change sign at that point.

For example, sketch $f(x)=x^3+x+1$.

19. Absolute Maxima and Minima

If $f(x_c)\geq f(x)$ for all *x* in the domain of *f*, then $f(x_c)$ is called the **absolute maximum value** of *f*.

If $f(x_c)\leq f(x)$ for all *x* in the domain of *f*, then $f(x_c)$ is called the **absolute minium value** of *f*.

Absolute extrema (if they exist) must always occur at critical point or at boundary points.

Procedures to fine absolute extrema on a closed interval

19.1 Find all critical point $x_{c,i}$ in the interval $[a,b]$. Differentiate with respect to (w.r.t.) *x* and equate to 0, i.e. $f'(x_{c,i})=0$.

19.2 Determine the critical values $f(x_{c,i})$ in the interval (a,b).

19.3 Find boundary value $f(a)$ and $f(b)$.

19.4 Compare $f(x_{c,i})$, $f(a)$ and $f(b)$ to find absolute minimum and maximum.

20. Asymptotes

20.1 Vertical asymptote (Any value of *x* for which *f*(*x*) goes toward $\pm\infty$.)

For example, rational function

$$f(x)=\frac{P(x)}{Q(x)}$$

$P(x)$ and $Q(x)$ are polynomial function. The line $x=a$ is a vertical asymptote for the graph of $f(x)$ if and only if $Q(x)=0$ and $P(x)\neq 0$.

20.2 Horizontal asymptote (Value of *f*(*x*) as *x* goes toward $\pm\infty$.)

If $\lim_{x\to\pm\infty} f(x)=b$, then $y=b$ is the horizontal asymptote.

21. Guild to Curve Sketching

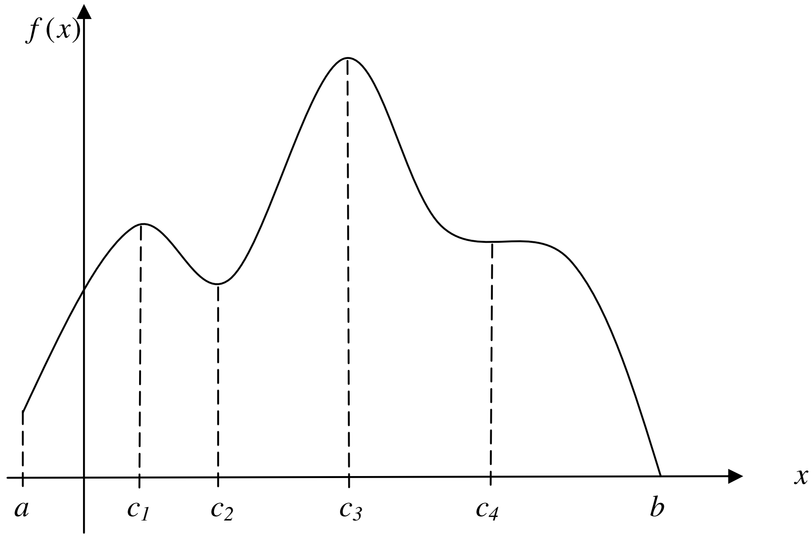
21.1 Look at **the function *f*(*x*) itself** to find *x*- or *y*- intercepts, vertical or horizontal asymptotes.

21.2 Look at **the first derivative** $f'(x)$ to find slop. It may be positive/negative for all values of *x*, so the function is increasing/decreasing as *x* is increasing. Otherwise, find the critical points $f'(x_c)=0$ and check the range where the graph is sloping up or down.

21.3 Look at **the second derivative** $f''(x)$ to see the curve behaviour i.e. check the nature of all critical points, check the range where the graph is concave up or down.

Note: Depending on the function, we may not need to do all of these steps to sketch the curve.

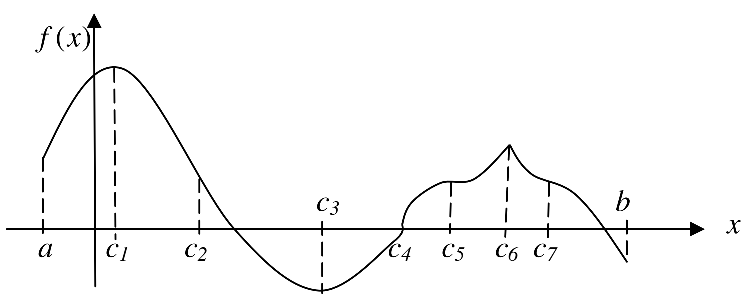
Ex. 1



The function above has ____ critical points which are relative maximum at ____ and ____, relative minimum at ____, and critical point which neither a maximum, nor minimum at ____. This is called ____________________________. The absolute maximum of the function over the range $[a,b]$ is at ____. The absolute minimum at _____ which is not a critical point.

Note that there may not be an absolute maximum and/or minimum if there are value of x that make $f(x) = \pm\infty$.

Ex. 2 Identify the points or intervals on the x axis that produce the indicated behaviour.



$f(x)$ is increasing	______________________________
$f'(x) < 0$	______________________________
Graph of f is concave down	______________________________
Local minima	______________________________
Absolute maxima	______________________________
$f'(x)$ appears to be 0	______________________________
$f'(x)$ is not unique	______________________________
Inflection points	______________________________