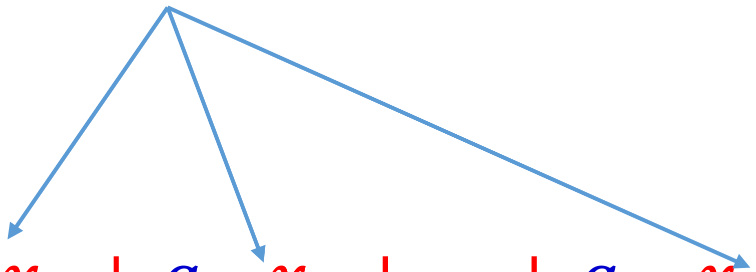
System of linear equations: exact identified system



N-Unknown variables

Equation 1: $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = d_1$

Equation 2: $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = d_2$

Equation N: $a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N = d_N$

$$Ax = d$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \\ d_N \end{bmatrix}$$

A = matrix of coefficients that are associated with unknown variables

d = matrix of coefficients that are independent of unknown variables

x = vector of unknown variables

= endogenous variables

Example: putting system of equations in $Ax = d$ form

- System of equations: 3 equations / 3 unknown.

$$\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}$$

$$Ax = d$$

$$\begin{array}{cccc} A & x & & d \\ \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} & = & \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} \\ (3 \times 3) & (3 \times 1) & & (3 \times 1) \end{array}$$

Matrix that collects all coefficients associated with each x.

Application 1

- Using the form $Ax = d$, write the following market demand model in the matrix form.

$$Qd = Qs$$

$$Qd = a - bP + cY$$

$$Qs = e + fP + gW$$

when a, b, c, e, f, g are parameters. Y = income and W = weather condition.

3 x 3 representation

$Qd = Qs$

$Qd = a - bP + cY$

$Qs = e + fP + gW$

$Qd - Qs + 0*P = 0$

$Qd + 0*Qs + bP = a + cY$

$0*Qd + Qs - fP = e + gW$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -f \end{bmatrix} \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ a + cY \\ e + gW \end{bmatrix}$$

2 x 2 representation: an alternative version

$Qd = Qs$

$Qd = a – bP + cY$

$Qs = e + fP + gW$

$Qd = Qs = Q$ (single Q!)

$Q = a – bP + cY$

$Q = e + fP + gW$

2 x 2 system equivalent to 3 x 3 one

$Q + bP = a + cY$

$Q - fP = e + gW$

$$\begin{bmatrix} 1 & b \\ 1 & -f \end{bmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} a + cY \\ e + gW \end{pmatrix}$$

Application 2

- National income model in the matrix form

$$y = c + I_0 + G_0$$

$$C = a + bY$$

$$\begin{aligned} Y \quad - \quad C &= I_0 + G_o \\ -bY \quad + \quad C &= a \end{aligned}$$

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{pmatrix} Y \\ C \end{pmatrix} = \begin{pmatrix} I_0 + G_o \\ a \end{pmatrix}$$

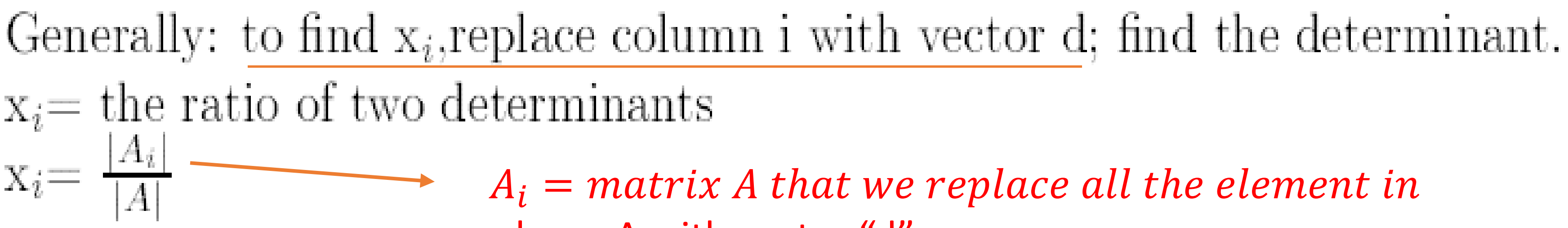
Solution for a system of linear equations

- Inverse matrix method
  - Find the inverse of the coefficient matrix associated with the unknown variables. (A)
  - Then, premultiply the inverse of coefficient matrix to both sides of the system.
  - $Ax = d \Rightarrow x = A^{-1}d$

Solution for a system of linear equations

- Cremer’s rule

Ax = d



Generally: to find x_i,replace column i with vector d; find the determinant.

x_i= the ratio of two determinants

$$x_i = \frac{|A_i|}{|A|}$$

A_i = *matrix A that we replace all the element in* column A with vector “d”

- Works only when system is just-identified. That is, A is the square matrix.

Determinant

- Determinant: a defined operation on "square" matrix that results in a scalar value outcome.
- Consider first 2 x 2 case. $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = k$
- A matrix whose determinant is equal to zero is called a singular matrix.

Determinant

Example 1

$$A=\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$|A| = (3 \times 5) - (1 \times 2) = 13 \qquad \{\text{Non-singular}\}$$

Example 2

$$B=\begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$$

$$|B| = (2 \times 24) - (6 \times 8) = 0 \qquad \{\text{Singular}\}$$

Determinant

- Next, 3 x 3 case.

Given A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

then

$$|A| = (a_1 b_2 c_3) + (a_2 b_3 c_1) + (b_1 c_2 a_3) - (a_3 b_2 c_1) - (a_2 b_1 c_3) - (b_3 c_2 a_1)$$

Cross-diagonals

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Determinant

- How about higher dimensions?
- The method is called Laplace expansion.
- Two concepts needed:
  - Minors
  - Cofactors

Determinant

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

- Minor

$|M_{ij}| \equiv$ is the subdeterminant from deleting the i-th row and the j-th column.

Determinants of sub-matrix
of the original matrix

- Example

$$|M_{11}| = \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} \qquad M_{21} \equiv \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Determinant

- Cofactors:

$$C_{ij} = (-1)^{i+j} \left| M_{ij} \right|$$

Determinant

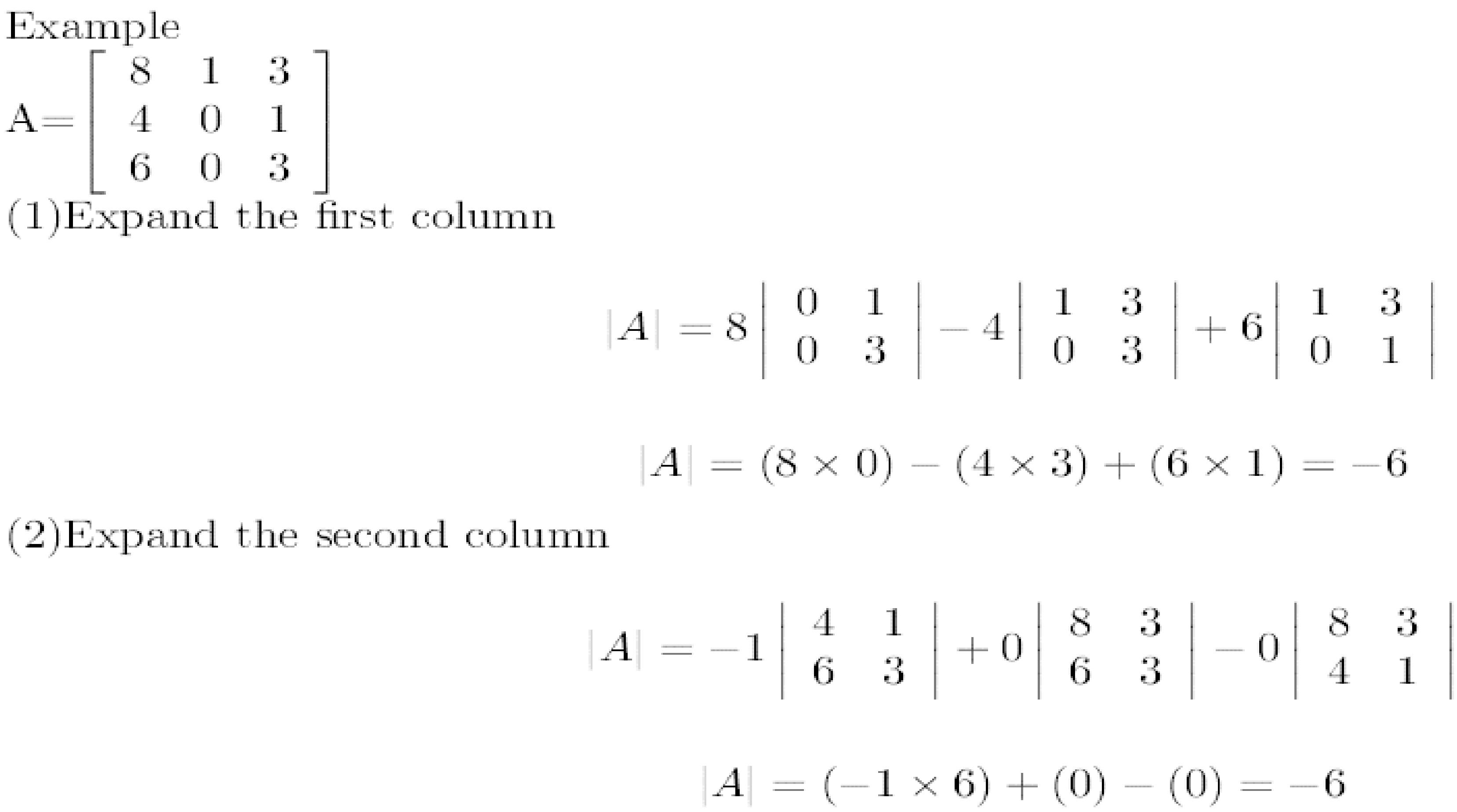
- Laplace expansion (along certain column/row)

$$A = \begin{array}{c} \begin{array}{ccc} \downarrow & \phantom{a_{12}} & \phantom{a_{13}} \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

$$|A| = a_{11}\,|C_{11}| + a_{21}\,|C_{21}| + a_{31}\,|C_{31}| = \sum_{i=1}^{3} a_{i1}\,|C_{i1}|$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Determinant



Example

$$A=\begin{bmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{bmatrix}$$

(1)Expand the first column

$$|A| = 8\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$|A| = (8 \times 0) - (4 \times 3) + (6 \times 1) = -6$$

(2)Expand the second column

$$|A| = -1\begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + 0\begin{vmatrix} 8 & 3 \\ 6 & 3 \end{vmatrix} - 0\begin{vmatrix} 8 & 3 \\ 4 & 1 \end{vmatrix}$$

$$|A| = (-1 \times 6) + (0) - (0) = -6$$

Inverse matrix

- Determinant of A and Adjoint matrix of A.

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

- What is the Adjoint matrix of A?
  - Transpose of the cofactor matrix of A.
  - Cofactor matrix of A is a matrix whose elements are the cofactors of the elements of A.

Inverse matrix

- Example

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{then } C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

- Example

$$\mathrm{A} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$$

Example

- Solve for the solution of below system, using the inverse matrix.

$4x + 3y - 28 = 0.$

$2x + 5y - 42 = 0.$

Solution

4x + 3y - 28 = 0.

2x + 5y – 42 = 0

$$\underbrace{\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}}_{det=14}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}28\\42\end{bmatrix}$$

$$cof\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}=\begin{bmatrix}5 & -2\\-3 & 4\end{bmatrix}$$

$$adj\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}=cof(A)^T=\begin{bmatrix}5 & -2\\-3 & 4\end{bmatrix}^T$$

$$=\begin{bmatrix}5 & -3\\-2 & 4\end{bmatrix}$$

$$\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}^{-1}=\frac{adj(A)}{\det(A)}=\frac{\begin{bmatrix}5 & -3\\-2 & 4\end{bmatrix}}{14}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\frac{\begin{bmatrix}5 & -3\\-2 & 4\end{bmatrix}}{14}\begin{bmatrix}28\\42\end{bmatrix}=\begin{bmatrix}5 & -3\\-2 & 4\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\8\end{bmatrix}\ \#$$

Example

- Example: solve for the equilibrium solution by using Cramer's rule.

$$Q^d = 10 - P$$

$$Q^s = P - 2$$

$Ax = d$ as the matrix representation of linear economic model

$$Ax = d$$

A = coefficients matrix (associated to x.)

x = Unknown endogenous variable

d = all the remaining

Note: putting all everything about endogenous variables on the left, the rest on the right.

Example

What is x? → $x = \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix}$

$$Q^d + P = 10$$

$Q^d = Q^s$ →

$$Q^s - P = -2$$

$$Q^d - Q^s = 0$$

Example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$d = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix}$$

Notice that $Q^d = Q^s$, and thus must be equal to Q

- Having imposed $Q^d = Q^s = Q$, we can reduce the 3-by-3 system into two-by-two system.
- Easier to solve.

$$Q^d = 10 - P$$

$$Q^s = P - 2$$

$$Q^d = Q^s = Q$$

⇨

$$Q + P = 10$$

$$P - Q = 2$$

Solution for a system of linear equations

$$\begin{array}{cccc} A & x & = & d \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} Q \\ P \end{bmatrix} & = & \begin{bmatrix} 10 \\ 2 \end{bmatrix} \end{array}$$

$$|A| = (1)(1) - (-1)(1) = 2$$

$$Q^e = \frac{\begin{vmatrix} 10 & 1 \\ 2 & 1 \end{vmatrix}}{2} = \frac{10-2}{2} = 4$$

$$P^e = \frac{\begin{vmatrix} 1 & 10 \\ -1 & 2 \end{vmatrix}}{2} = \frac{2-(-10)}{2} = 6$$

Q is associated with the first column of coefficient matrix, so replace the first column with “d”.

P is associated with the second column of coefficient matrix, so replace the second column with “d”.

Example

Consider a simple macroeconomic model.

$C = a + bY_d; \quad 0 < b < 1$

$I = I_a + iY; \quad 0 < i < 1$

$G = G_0$

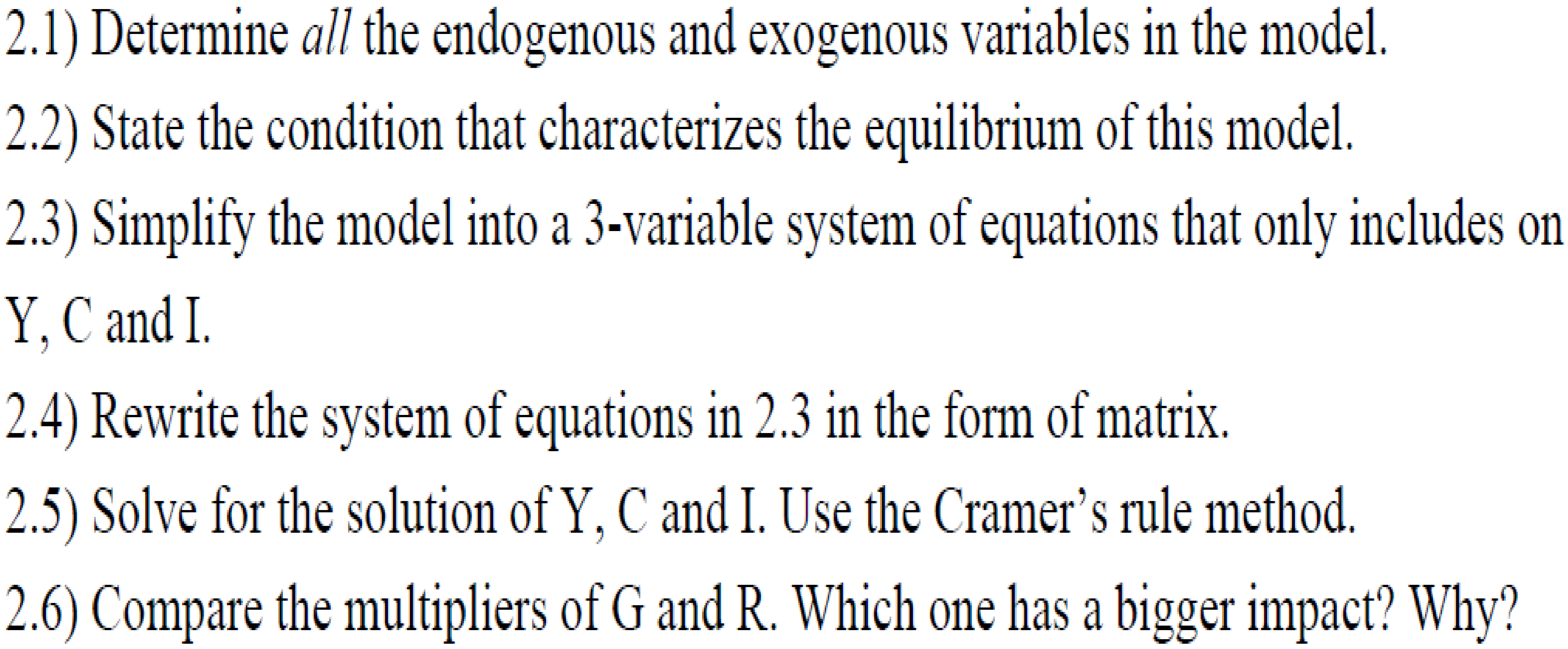
$T = T_0 + tY; \quad 0 < t < 1$

$R = R_0$

$Y_d = Y - T + R$

where R is the government transfer and G is the government purchase. All the remainings are defined as usual.

Solution for a system of linear equations



2.1) Determine *all* the endogenous and exogenous variables in the model.

2.2) State the condition that characterizes the equilibrium of this model.

2.3) Simplify the model into a 3-variable system of equations that only includes on Y, C and I.

2.4) Rewrite the system of equations in 2.3 in the form of matrix.

2.5) Solve for the solution of Y, C and I. Use the Cramer's rule method.

2.6) Compare the multipliers of G and R. Which one has a bigger impact? Why?

Endo v.s. Exo

- Endo: $Y, Y_d, C, I\ , T$
- Exo: $\mathrm{G}\ \ and\ \ R$

- 6 equations, so one missing equation?

→ Equilibrium condition: Y = C + I + G

Simplify the system into 3-by-3 matrix

$$Y = C + I + G_0$$

$$I = I_a + iY$$

$$\begin{aligned} C &= a + bY_d \\ &= a + b(Y - T_0 - tY + R_0) \\ &= a + b(1 - t)Y + b(R_0 - T_0) \end{aligned}$$

Rearranging terms to make the system compatible with Ax = d form

$$Y = C + I + G_0 \longrightarrow Y - C - I = G_0$$

$$I = I_a + iY \longrightarrow I - iY = I_a$$

$$\begin{aligned} C &= a + bY_d \\ &= a + b(Y - T_0 - tY + R_0) \\ &= a + b(1 - t)Y + b(R_0 - T_0) \longrightarrow C - b(1 - t)Y = a + b(R_0 - T_0) \end{aligned}$$

Represent the system in the matrix form: Ax = d

$$Y - C - I = G_0$$

$$I - iY = I_a$$

$$C - b(1-t)Y = a + b(R_0 - T_0)$$

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 \\ -i & 0 & 1 \\ -b(1-t) & 1 & 0 \end{bmatrix}}_{det:=1-i-b(1-t)} \begin{bmatrix} Y \\ C \\ I \end{bmatrix} = \begin{bmatrix} G_0 \\ I_a \\ a + b(R_0 - T_0) \end{bmatrix}$$

Cramer's rule

$$Y^* = \frac{\begin{vmatrix} G_0 & -1 & -1 \\ I_a & 0 & 1 \\ a+b(R_0-T_0) & 1 & 0 \end{vmatrix}}{1-i-b(1-t)}$$

$$Multiplier\ of\ G_0\ \ on\ \ Y^* = \frac{1}{1-i-b(1-t)}$$

$$Multiplier\ of\ R_0\ \ on\ \ Y^* = \frac{b}{1-i-b(1-t)}$$

$$C^* = \frac{\begin{vmatrix} 1 & G_0 & -1 \\ -i & I_a & 1 \\ -b(1-t) & a+b(R_0-T_0) & 0 \end{vmatrix}}{1-i-b(1-t)}$$

$$Multiplier\ of\ G_0, R_0\ \ on\ \ C^*\ and\ I^* = ?????$$

$$I^* = \frac{\begin{vmatrix} 1 & -1 & G_0 \\ -i & 0 & I_a \\ -b(1-t) & 1 & a+b(R_0-T_0) \end{vmatrix}}{1-i-b(1-t)}$$