Solution: Assignment 5

1. (Optional) Let $X = \{2, 3, 4\}$ and $Y = \{1, 5\}$. Define relations
$f : X \to Y$ by $f = \{(x, y) \in X \times Y | x > y\}$,
$g : X \to Y$ by $g = \{(x, y) \in X \times Y | x = 2y\}$, and
$h : X \to Y$ by $h = \{(x, y) \in X \times Y | y = x^2\}$.

   (a) List all the elements of the Cartesian product $X \times Y$.

   (b) List all the elements of the relations f, g, and h.

   (c) Determine which of the relations f, g, h is a function from X to Y. Explain your answer.

   (d) What is the inverse image of 1 under f?

   (e) What is the inverse image of 5 under f?

<u>**Solution:**</u>

(a) List all the elements of the Cartesian product $X \times Y$.

$$X \times Y = \{(2, 1), (2, 5), (3, 1), (3, 5), (4, 1), (4, 5)\}$$

(b) List all the elements of the relations f, g, and h.

$$f = \{(2, 1), (3, 1), (4, 1)\}.$$
$$g = \{(2, 1)\}$$
$$h = \{\} = \emptyset$$

(c) Determine which of the relations f, g, h is a function. Explain your answer.

-The relation f is a function because all the elements in the domain X get mapped to an element in Y and each of them gets mapped only once (even though the element 1 in Y gets mapped more than once).

-The relation g is not a function from X to Y, since there are some element in X doesn't get mapped to Y(i.e. $x = 3, 4$).

-The relation h is not a function from X to Y, since none of the elements in X gets mapped (i.e. $x = 2, 3, 4$) to Y.

(d) What is the inverse image of 1 under f?

$$\{x \in X | (x, 1) \in f\} = \{2, 3, 4\}$$

(e) What is the inverse image of 5 under f?

$$\{x \in X | (x, 5) \in f\} = \emptyset$$

2. Let $A = \{1, 2, 3\}$ and let $\mathcal{P}(A)$ be the set of all subsets of the set A and let X be a subset of $\mathcal{P}(A)$, which defined as
$$X = \{x \in \mathcal{P}(A) |\ x \cap \{1\} = \emptyset\}.$$
Define relations r and s as
$$r = \{(x, y) \in X \times \mathcal{P}(A) \mid\ x \text{ and } y \text{ have the same number of elements } \},$$
$$s = \{(u, v) \in \mathcal{P}(A) \times X \mid (v, u) \in r\}.$$
   (a) Determine the sets $\mathcal{P}(A)$ and X.
   (b) Draw arrow diagrams of r and s.
   (c) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.
   (d) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.

Solution:

(a) Determine the sets $\mathcal{P}(A)$ and X.

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

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Consider $X = \{x \in \mathcal{P}(A) |\ x \cap \{1\} = \emptyset\}$. Notice that, for $x \in \mathcal{P}(A)$, $\{1\} \cap x = \emptyset$ when the set x does not contain the element 1, e.g.
if $x = \emptyset$, $\{1\} \cap x = \emptyset$ and $x = \emptyset \in X$;
if $x = \{1\}$, $\{1\} \cap x \neq \emptyset$ and $x = \{1\} \notin X$;
if $x = \{2\}$, $\{1\} \cap x = \emptyset$ and $x = \{2\} \in X$;
if $x = \{1, 2\}$, $\{1\} \cap x \neq \emptyset$ and $x = \{2\} \notin X$;
if $x = \{2, 3\}$, $\{1\} \cap x = \emptyset$ and $x = \{2, 3\} \in X$;
if $x = \{1, 2, 3\}$, $\{1\} \cap x \neq \emptyset$ and $x = \{1, 2, 3\} \notin X$, etc.
Hence,
$$X = \{x \in \mathcal{P}(A) |\ x \cap \{1\} = \emptyset\} = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$$

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(b) Draw an arrow diagrams of r and s.

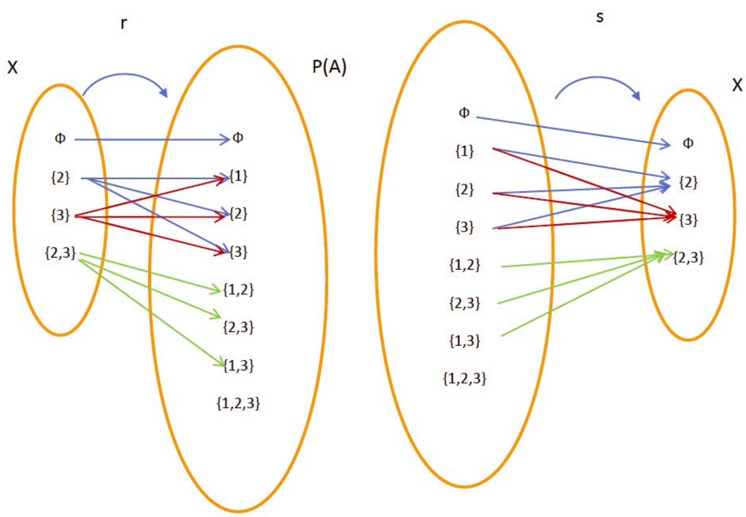

Figure 1: Problem 2(b)

(c) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.
No, r is a not a function because the element $\{2\} in X$ gets mapped to $\{1\}, \{2\}, \{3\} \in \mathcal{P}(A)$(i.e. it gets mapped more than once and similarly for $\{2\}, \{3\}, \{2, 3\} \in X$). ■

(d) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.
No, s is a not a function because the elements $\{1\}, \{2\}, \{3\}$ in the domain $\mathcal{P}(A)$ get mapped more than once. Also the element $\{1, 2, 3\}$ in the domain does not get mapped to any element in X.

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3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (y/2, x - 1) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

(a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Is H one-to-one? Prove or give a counterexample.
Answer: Yes, H is one-to-one. Let $(x_1, y_1), (x_2, y_2)$ be some elements in the domain $\mathbb{R} \times \mathbb{R}$. We want to show that if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$.
Suppose $H(x_1, y_1) = H(x_2, y_2)$. Then

$$(y_1/2, x_1 - 1) = (y_2/2, x_2 - 1)$$

or equivalently, $y_1/2 = y_2/2$ and $x_1 - 1 = x_2 - 1$. I.e.,

$$y_1/2 = y_2/2 \quad \Rightarrow \quad y_1 = y_2 \quad \text{and} \quad x_1 - 1 = x_2 - 1 \quad \Rightarrow \quad x_1 = x_2$$

That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$ and therefore, H is one-to-one.

(b) Is H onto? Prove or give a counterexample.
Answer: Yes, H is not onto. Notice that if we pick (u, v) from the co-domain $\mathbb{R} \times \mathbb{R}$ and suppose that there is (x, y) in the domain such that $H(x, y) = (u, v)$, then

$$(u, v) = H(x, y) = (y/2, x - 1\ )$$

or we must have

$$u = y/2 \quad \Rightarrow \quad y = 2u \in \mathbb{R}$$

and

$$v = x - 1 \quad \Rightarrow \quad x = v + 1 \in \mathbb{R}.$$

That is, for a given (u, v) from the co-domain $\mathbb{R} \times \mathbb{R}$ we can use $(x, y) = (v + 1, 2u)$ **in the domain** $\mathbb{R} \times \mathbb{R}$ so that

$$H(x, y) = H\,(v + 1, 2u) = (2u/2, (v + 1) - 1\ ) = (u, v).$$

Therefore, H is onto.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.
Answer: Yes.
From (a) and (b), since H is both one-to-one and onto, then H bijective.
To find the inverse function, H^{-1}, of H, we recall from the definition

$$H^{-1}(u, v) = (x, y) \quad \Leftrightarrow \quad H(x, y) = (u, y).$$

Since we have from (b) that

$$H\,(v + 1, 2u) = (u, v),$$

and hence the inverse function $H^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is given by

$$H^{-1}(u, v) = (v + 1, 2u)\,.$$

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4. (Optional) Define functions f and g as follows:
$f = \{(1, 10), (3, 30), (5, 50)\}$ and
$g = \{(10, k), (20, \ell), (30, m), (40, n), (50, t)\}$.

(a) Determine the domain and range for each of functions f and g.

(b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.

(c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.

Solution:

(a) Determine the domain and range for each of functions f and g.

$D_f = \{1, 3, 5\}$, $R_f = \{10, 30, 50\}$,
$D_g = \{10, 20, 30, 40, 50\}$, $R_g = \{k, \ell, m, n, t\}$

(b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.
$\underline{g \circ f}$: Since $R_f \cap D_g = \{10, 30, 50\} \neq \emptyset$, we can construct $g \circ f$:
$(g \circ f)(1) = g(f(1)) = g(10) = k$
$(g \circ f)(3) = g(f(3)) = g(30) = m$
$(g \circ f)(5) = g(f(5)) = g(50) = t$.

That is, $g \circ f = \{(10, k), (30, k), (50, t)\}$.
The domain is $\{10, 30, 50\}$ and the range is $\{k, m, t\}$.
$\underline{f \circ g}$: Since $R_g \cap D_f = \{k, \ell, m, n, t\} cap \{1, 3, 5\} = \emptyset$, we cannot construct $f \circ g$.

(c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.
First notice that g^{-1} is a function since g is bijective.

$$g^{-1} = \{(k, 10), (\ell, 20), (m, 30), (n, 40), (t, 50)\}.$$

That is , $D_{g^{-1}} = \{k, \ell, m, n, t\}$ and $R_{g^{-1}} = \{10, 20, 30, 40, 50\}$.

$\underline{g \circ g^{-1}}$: Since $R_{g^{-1}} \cap D_g = \{10, 20, 30, 40, 50\} \neq \emptyset$, we can construct $g \circ g^{-1}$:
$(g \circ g^{-1})(k) = g(g^{-1}(k)) = g(10) = k$
$(g \circ g^{-1})(\ell) = g(g^{-1}(\ell)) = g(20) = \ell$
$(g \circ g^{-1})(m) = g(g^{-1}(m)) = g(30) = m$
$(g \circ g^{-1})(n) = g(g^{-1}(n)) = g(40) = n$
$(g \circ g^{-1})(t) = g(g^{-1}(t)) = g(50) = t$.
That is,

$$g \circ g^{-1} = \{(k, k), (\ell, \ell), (m, m), (n, n), (t, t)\}.$$

That is, $D_{g \circ g^{-1}} = R_{g \circ g^{-1}} = \{k, \ell, m, n, t\}$.
$\underline{g^{-1} \circ g}$: Since $D_{g^{-1}} \cap R_g = \{k, \ell, m, n, t\} \neq \emptyset$, we can construct $g \circ g^{-1}$:
$(g^{-1} \circ g)(10) = g^{-1}(g(10)) = g^{-1}(k) = 10$
$(g^{-1} \circ g)(20) = g^{-1}(g(20)) = g^{-1}(\ell) = 20$
$(g^{-1} \circ g)(30) = g^{-1}(g(30)) = g^{-1}(m) = 30$
$(g^{-1} \circ g)(40) = g^{-1}(g(40)) = g^{-1}(n) = 40$
$(g^{-1} \circ g)(50) = g^{-1}(g(50)) = g^{-1}(t) = 50$.
That is,

$$g^{-1} \circ g = \{(10, 10), (20, 20), (30, 30), (40, 40), (50, 50)\}.$$

That is, $D_{g^{-1} \circ g} = R_{g^{-1} \circ g} = \{10, 20, 30, 40, 50\}$.

5. Define

$$g(x) = \sqrt{x^2 + 1} \qquad \text{and} \qquad f(x) = \begin{cases} \frac{x}{2}, & x < 2 \\ x - 1, & x \geq 2. \end{cases}$$

(a) Find the domain and range for each of the functions g and f.

(b) Construct the composite functions $f \circ g$, and $g \circ f$ (if possible). Determine the domains for these composite functions.

(c) (Optional) Is the function f bijective? If so, find the **inverse function** of f. Justify your answer.

Solution:

(a) Find the domain, co-domain, and range for each of the functions g and f.

To find the domain of g, notice that we must have $x^2 + 1 > 0$ for any real number x. That is, $x \in (-\infty, \infty) = \mathbb{R}$. To find the range of g,

$$x^2 \geq 0 \quad \Rightarrow \quad x^2 + 1 \geq 1 \quad \Rightarrow \quad \sqrt{x^2+1} \geq \sqrt{1} = 1 \quad \Rightarrow \quad \sqrt{x^2+1} \geq 1 \quad \Rightarrow \quad g(x) \geq 1$$

That is,

$$D_g = \mathbb{R}, \qquad \text{and} \qquad R_g = [1, \infty).$$

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To find the domain of f, notice that $x/2$ is well-defined for $x \in (-\infty, 2)$ and $x - 1$ is well-defined for $x \in [2, \infty)$.
Hence the domain $D_f = (-\infty, 2) \cup [2, \infty) = \mathbb{R}$.
To find the range of f, we consider 2 cases.

(i) Suppose $x \in (-\infty, 2)$.
$x < 2 \quad \Rightarrow \quad x/2 < 2/2 \quad \Rightarrow \quad f(x) < 1 \ \Rightarrow \ f(x) \in (-\infty, 1)$.

That is, $f(x) \in (-\infty, 1)$ when $x < 2$.

(ii) Suppose $x \in [2, \infty)$.
$x \geq 2 \quad \Rightarrow \quad x - 1 \leq 2 - 1 = 1 \ \Rightarrow \ f(x) \in [1, \infty)$.

That is, $f(x) \in [1, \infty) = \mathbb{R}$ when $x \geq 2$.

Hence from (i) and (ii), $f(x) \in (-\infty, 1) \cup [1, \infty) = \mathbb{R}$ and the range of f is $R_f = \mathbb{R}$.
That is,

$$D_g = \mathbb{R}, \qquad \text{and} \qquad R_g = [1, \infty).$$
$$D_f = \mathbb{R}, \qquad \text{and} \qquad R_f = \mathbb{R}.$$

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(b) Construct the composite functions $f \circ g$, and $g \circ f$ (if possible). Determine the domains for these composite functions.
$\underline{f \circ g}$: Notice that $R_g \cap D_f = [1, \infty) \cap \mathbb{R} \neq \emptyset$. So, we can construct $f \circ g = f(g(x))$ by considering two cases.

(I) $g(x) = \sqrt{x^2+1} < 2$, which occurs when $x^2 + 1 < 4$

$$x^2 - 3 < 0 \quad \Rightarrow \quad -\sqrt{3} < x < \sqrt{3} \quad \Leftrightarrow \quad x \in (-\sqrt{3}, \sqrt{3}) \subseteq D_g.$$

In this case,

$$f \circ g = f(g(x)) = g(x)/2 = \frac{\sqrt{x^2+1}}{2}.$$

(II) $g(x) = \sqrt{x^2+1} \geq 2$, which occurs when $x^2+1 \geq 4$

$$x^2 - 3 \geq 0 \quad \Rightarrow \quad x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty) \subseteq D_g.$$

In this case,

$$f \circ g = f(g(x)) = g(x) + 1 = \sqrt{x^2+1} + 1.$$

From (I) and (II),

$$(f \circ g)(x) = \begin{cases} \frac{\sqrt{x^2+1}}{2}, & x \in (-\sqrt{3}, \sqrt{3}) \\ \sqrt{x^2+1} + 1, & x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty). \end{cases}$$

To find the domain of $f \circ g$, i.e. $D_{f \circ g}$, since $R_g \subseteq D_f$, then $D_{f \circ g} = D_g = \mathbb{R}$. ■

$\underline{g \circ f}$: Notice that $R_f \cap D_g = \mathbb{R} \cap \mathbb{R} \neq \emptyset$. So, we can construct $g \circ f$: for $x \in D_f = \mathbb{R}$,

$$(g \circ f)(x) = g(f(x)).$$

(I) For $x < 2$, $f(x) = x/2 \in D_g = \mathbb{R}$ and

$$(g \circ f)(x) = g(f(x)) = g(x/2) = \sqrt{(x/2)^2 + 1}.$$

(II) For $x \geq 2$, $f(x) = x - 1 \in D_g = \mathbb{R}$ and

$$(g \circ f)(x) = g(f(x)) = g(x-1) = \sqrt{(x-1)^2 + 1}.$$

That is, from (I) and (II)

$$(g \circ f)(x) = \begin{cases} \sqrt{(x/2)^2 + 1}, & x < 2 \\ \sqrt{(x-1)^2 + 1}, & x \geq 2. \end{cases}$$

and the domain is $D_{g \circ f} = [-3, 0) \cup (0, 1) \cup [1, 3) = [-3, 0) \cup (0, 3)$. ■

To find the domain of $g \circ f$, i.e. $D_{g \circ f}$, since $R_f \subseteq D_g$ ($R_f = D_g = \mathbb{R}$), then $D_{g \circ f} = D_f = \mathbb{R}$. ■

(c) (Optional)Is the function f bijective? If so, find the **inverse function** of f. Justify your answer.

[Brief answer] Yes. From (i) and (ii) in part (a):

(i) For $x \in (-\infty, 2)$, $f(x) = x/2 \in (-\infty, 1)$.

(ii) For $x \in [2, \infty)$, $f(x) = x - 1 \in [1, \infty)$.

$$f^{-1}(y) = \begin{cases} 2y, & y < 1 \\ y + 1, & y \geq 1. \end{cases}$$

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Additional Optional Problems

1. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Find $f \circ g$, $g \circ f$, and determine whether or not $f \circ g = g \circ f$ for the given formulas for f and g. Compute $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (a) $f(x) = \frac{x}{\sqrt{x^2+1}}, \quad g(x) = x^3 + 1.$

   (b) $f(x) = x^5, \quad g(x) = x^{1/5}.$

2. Let $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$ be a function defined by $f(x) = \frac{2x+1}{1-x}$.

   (a) Show that f is bijective.

   (b) Determine the inverse function f^{-1}.

   (c) Compute $f \circ f$ and $f \circ f^{-1}$.

3. Define $f : \mathbb{Z} \to \mathbb{Z}$ by the rule $f(n) = 2 - 3n$, for all integers n.

   (i) Is f one-to-one? Prove or give a counterexample.

   (ii) Is f onto? Prove or give a counterexample.

4. Define $G : \mathbb{R} \to \mathbb{R}$ by the rule $G(x) = 2 - 3x$ for all real numbers x. Is G onto? Prove or give a counterexample.

5. Define $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by $f(x) = \frac{x+1}{x}$, for all real numbers $x \neq 0$. Determine whether or not f is one-to-one and justify your answer.

6. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{x^2+1}$, for all real numbers x. Determine whether or not f is one-to-one and justify your answer.

7. Define $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows: $G(x, y) = (2y, -x)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

   a. Is G one-to-one? Prove or give a counterexample.

   b. Is G onto? Prove or give a counterexample.

8. You work forty hours a week at a furniture store. You receive a \$220 weekly salary, plus a 3% commission on sales over \$5000. Assume that you sell enough this week to get the commission. Given the functions $f(x) = 0.03x$ and $g(x) = x - 5000$, which of $(f \circ g)(x)$ and $(g \circ f)(x)$ represents your commission?

   Answer:
   Consider $f(x) = 0.03x$, for $x \geq 0$ and $g(x) = x - 5000$.

   Then $(f \circ g)(x) = f(g(x))$ represents the **commission** because $g(x)$ gives the amount of money that exceeds \$5000 and we will calculate the commission from 3% of this amount $g(x)$. ■

9. You make a purchase at a local hardware store, but what you've bought is too big to take home in your car. For a small fee, you arrange to have the hardware store deliver your purchase for you. You pay for your purchase, plus the sales taxes, plus the fee. The taxes are 7.5% and the fee is \$20.

   (i) Write a function $t(x)$ for the total, after taxes, on the purchase amount x. Write another function f(x) for the total, including the delivery fee, on the purchase amount x.

(ii) Calculate and interpret $(f \circ t)(x)$ and $(t \circ f)(x)$. Which results in a lower cost to you?

(iii) Suppose taxes, by law, are not to be charged on delivery fees. Which composite function must then be used?

Answer:

(i) Write a function $t(x)$ for the total, after taxes, on the purchase amount x. Write another function $f(x)$ for the total, including the delivery fee, on the purchase amount x.

Taxes function: $t(x) = 1.075x$
Fee Function: $f(x) = x + 20$ ■

(ii) Calculate and interpret $(f \circ t)(x)$ and $(t \circ f)(x)$. Which results in a lower cost to you?

Notice that $(f \circ t)(x) = f(t(x))$ is the cost when the taxes are calculated **before** applying the fee.

$$(f \circ t)(x) = f(t(x)) = f(1.075x) = 1.075x + 20$$

and $(t \circ f)(x) = t(f(x))$ is the cost when the taxes are calculated **before** applying the fee.

$$(t \circ f)(x) = t(f(x)) = f(x + 20) = 1.075(x + 20) = 1.075x + 21.5.$$

From above, we see that $(f \circ t)(x) = 1.075x + 20 < 1.075x + 21.5 = (t \circ f)(x)$ and therefore calculating taxes before combining with the fee has a lower cost. ■

(iii) Suppose taxes, by law, are not to be charged on delivery fees. Which composite function must then be used?

The composite function $(f \circ t)(x)$ must be used because taxes must be calculated before combining with the fee. ■

10. Your computer's screen saver is an expanding circle. The circle starts as a dot in the middle of the screen and expands outward, changing colors as it grows. With a twenty-one inch screen, you have a viewing area with a 10-inch radius (measured from the center diagonally down to a corner). The circle reaches the corners in four seconds. Express the area of the circle (discounting the area cut off by the edges of the viewing area) as a function of time t in seconds.

Answer:
Since the circle's leading edge covers ten inches, the radius is growing at a rate of 10 inches/ 4 seconds = 2.5 inches per second. Hence the radius can be written in terms of time as

$$r(t) = 2.5t.$$

Using the formula for the area of a circle,

$$A(r) = \pi r^2.$$

Then

$$A(t) = (A \circ r)(t) = A(r(t)) = A(2.5t) = \pi(2.5t) = 6.25\pi t^2.$$

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