7 Additional Topics In Integration

7.1 Integration by Parts

Integration by parts technique could become very useful if the integrand is the **product of two functions.** Starting from the product rule of differentiation,

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Integrating both sides of the above equation

$$f(x)g(x) + C_1 = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) + C_1 - \int f'(x)g(x)dx$$

Absorbing C_1 into the constant of integration for $\int f'(x)g(x)dx$. Hence,

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx} \qquad (1) \quad \text{Given!}$$

Some books, written (1) in slightly different form.

$$\int udv = uv - \int vdu$$

Hints: Suppose $H(x)$ is a product of two functions, to integrate $H(x)$, write it in the form of $f(x)g'(x)$. We need to choose $f(x)$ and $g'(x)$ very carefully so that we can calculate $f'(x)$ and $g(x)$, and we can **easily** integrate $\int f'(x)g(x)dx$. Sometimes the ***best choice*** for $f(x)$ and $g'(x)$ is not obvious. Insight into making a good choice will come only with ***practice*** and, of cause, ***trial and error***.

Integration by parts also works for definite integral.

$$\boxed{\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)dx}$$

Ex.1: Determine

(a) $\int xe^x dx$

(b) $\int \frac{3x}{\sqrt{4-x}} dx$

(c) $\int \frac{x}{(5x+2)^3} dx$

Ans: (a) Determine $\int xe^x dx$

xe^x is a product of two functions so it is possible to use integration by parts.

Let's $\quad f(x) = x \qquad g'(x) = e^x$

$\qquad f'(x) = 1 \qquad g(x) = \int e^x dx = e^x [+C_1]$ *Will add constant of integration at the end.*

Recall:
$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$
$$\int xe^x dx = (x)(e^x) - \int (1)(e^x)dx$$
$$\int xe^x dx = xe^x - e^x + C$$

Check:
$$\frac{d}{dx}(xe^x - e^x + C) = [(1)e^x + xe^x] - e^x + 0 = xe^x$$

Note: Selecting $f(x)$ and $g'(x)$ badly will complicate the problem!

For example, let's $\quad f(x) = e^x \qquad g'(x) = x$

$\qquad f'(x) = e^x \qquad g(x) = \frac{x^2}{2}$

Recall:
$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$
$$\int e^x x dx = e^x\left(\frac{x^2}{2}\right) - \int e^x\left(\frac{x^2}{2}\right)dx$$

$\int e^x\left(\frac{x^2}{2}\right)dx$ is in fact more complicated than the original integral $\int xe^x dx$.

Carefully choose $f(x)$ and $g'(x)$. Practice to gain more experience!

Ex.2: By using integration show that

(a) $\int e^{2x}x^2dx = \frac{1}{4}\left(2x^2 - 2x + 1\right)e^{2x} + C$

Hint: we will need to use integration by parts twice.

(b) $\int e^{2x}x^3dx = \frac{x^3e^{2x}}{2} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$.

Hint: we will need to use integration by parts three times.

Ex.3: Find $I = \int \ln x dx$. [Ans: $I = x[\ln x - 1] + C$]

Ex.4: Find $I = \int xe^{-x}dx$. [Ans: $I = -e^{-x}[1 + x] + C$]

Ex.5: Find $I = \int y^3 \ln y dy$. [Ans: $I = \frac{y^4 \ln y}{4} - \frac{y^4}{16} + C$]

Ex.6: If the marginal cost of producing x units is given by $C'(x) = 0.3x^2 + 2x$ and the fixed cost is 2,000 pounds, find the cost function $C(x)$ and the cost of producing 20 units.

[Ans: $C(x) = 0.1x^3 + x^2 + 2000$, $C(20) = 3,200$]

Ex.7: Find the revenue function $R(x)$ when the marginal revenue is $R'(x) = 400 - 0.4x$ and no revenue results at a 0 production level. What is the revenue at a production level of 1,000 units? [Ans: $R(x) = 400x - 0.2x^2$, $R(1,000) = 200,000$]

Ex.8: A radio program, *the PM meets the people*, is launching an aggressive advertising campaign in order to increase the number of daily listeners. The programme currently has 27,000 weekly listeners, and the PM expects the number of daily listeners, $S(t)$, to grow at the rate of $S'(t) = 60t^{\frac{1}{2}}$ listeners per week, where t is the number of weeks since the campaign began. How long should the campaign last if the PM wants the number of weekly listeners to grow to 41,000?

[Ans: 50 weeks]

Ex.9: The current weekly circulation of the magazine *Business Week* is 640,000 copies. Due to competition from a new magazine in the same field, the weekly circulation of *Business Week*, $B(t)$, is expected to decrease at the rate of $B'(t) = -6000t^{\frac{1}{3}}$ copies per week, where t is the time in weeks since the new magazine began publication. How long will it take for the circulation of *Business Week* to decrease to 460,000 copies per week?
[Ans: 16]

Ex.10: Bangjak Petroleum has been introducing Gasohol 95 at its stations since 2004 in preparation for the government's deadline of banning benzene 95 on 1st January 2006. The consumption has been growing at a rate given by

$$f'(t) = 0.004t + 0.62 \text{ million litres}$$

where t is years after 2004. At the end of 2005 (the 2nd year), the consumption is 2 million litres. Find $f(t)$ and estimate the consumption of Gasohol 95 in 2020.

[Ans: $f(t) = 0.002t^2 + 0.62t + 0.752$, 11.87]

Ex.11: The market research department for Tesco-Lotus supermarket chain has determined that for one store the marginal price $p'(x)$ at x tubes per week for a brand of sugar free gum is given by $p'(x) = -0.015e^{-0.01x}$. Find the price-demand equation if the weekly demand is 50 packs when the price of a tube is ฿2.35. Find the weekly demand when the price of a pack is ฿1.89. [Ans: 120 packs]

Ex.12: Thammasat University Press publishes x textbooks per week. The weekly marginal profit is given by

$$P'(x) = 165 - 0.1x \qquad 0 \le x \le 4{,}000$$

The TU press is currently manufacturing 1,500 books per week, but is planning to increase production. Find the change in the weekly profit if weekly production is increases to 1,600 books. [Ans: 1,000]

Ex.13: A game shop maintains records for each computer game it installs. Suppose that $C(t)$ and $R(t)$ represent the total accumulated costs and revenues (in thousands of bahts) respectively, t months after a particular game has been installed and that

$$C'(t) = 2 \qquad R'(t) = 9e^{-0.5t}$$

The value of t for which $C'(t) = R'(t)$ is called the **useful life** of the game.

(A) Find the useful life of the game to the nearest year. [Ans: 3]
(B) Find the total profit accumulated during the useful life of the game. [Ans: 7,984]

7.2 Integration by Table

There are some functions that cannot be integrated by substitution and by parts. In addition, there are some functions, for example $\int \frac{e^x}{x}\,dx$ that cannot be integrated at all. However, there are some functions that can be integrated using a technique called **Integration by Table**. A short table of integrals is listed on pages 472 and 473 of Hoffmann and Bradley, Calculus for Business, Economics, and the Social and Life Sciences, 9th edition.

Ex. 14: Find $\int \frac{1}{x(3x-6)}\,dx$.

Solution: Apply Formula 6 $\left(\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C \right)$ where $u = x, a = -6, b = 3$

$$\int \frac{1}{x(3x-6)}\,dx = \frac{1}{-6}\ln\left|\frac{x}{-6+3x}\right| + C = -\frac{1}{6}\ln\left|\frac{x}{3x-6}\right| + C$$

Ex. 15: Find $\int \frac{1}{6-3x^2}\,dx$.

Solution: If the coefficient of x^2 were 1 instead of 3, we will be able to use Formula 16 $\left(\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C \right)$ so we will first rearrange the integrand as

$$\frac{1}{6-3x^2} = \frac{1}{3}\left(\frac{1}{2-x^2}\right)$$

Hence using Formula 16 with $u = x, a = \sqrt{2}$

$$\int \frac{1}{3}\left(\frac{1}{2-x^2}\right)dx = \frac{1}{3}\int\left(\frac{1}{2-x^2}\right)dx = \left(\frac{1}{3}\right)\left(\frac{1}{2\sqrt{2}}\right)\ln\left|\frac{\sqrt{2}+x}{\sqrt{2}-x}\right| + C$$

$$= \frac{1}{6\sqrt{2}}\ln\left|\frac{\sqrt{2}+x}{\sqrt{2}-x}\right| + C$$

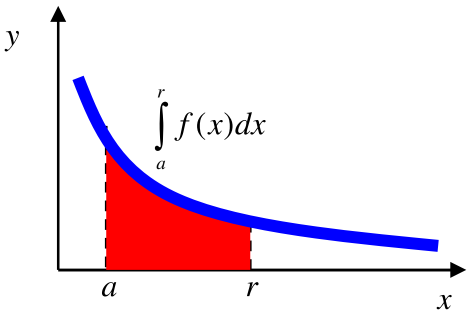
7.3 Summary of Integration Techniques

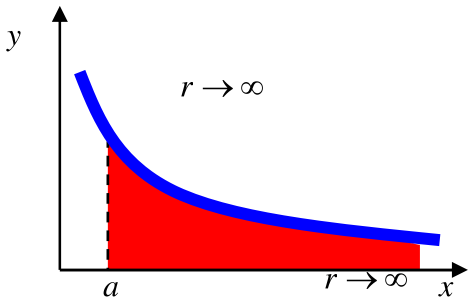
When you are facing an integration problem, try the following technique in order.

Techniques	**Notes**
(1) Integration by simple formulas	Try to arrange to get to the simple formats.
(2) Integration by substitution	This technique is useful when we can choose u so that we have $\left(\frac{du}{dx}\right)$ as part of the integrand.
(3) Integration by parts	This technique is useful when we have a product of two functions that we can choose $f(x)g'(x)$.
(4) Integration by advanced formulas	This is useful when we cannot use technique (1) – (3). We may need to rearrange the integrand into the form that is appropriated to use the formulas.

7.4 Improper Integrals

Improper integrals is when the limit is ∞ and/or $-\infty$. That is $\int_a^{+\infty} f(x)dx$





Area becomes $$\int_a^{\infty} f(x)dx = \lim_{r\to\infty} \int_a^r f(x)dx$$

If the limit **exists**, $\int_a^{\infty} f(x)dx$ is convergent or it converges. Hence, the area under the curve is finite and equal to the value of the limit $\left(\lim_{r\to\infty} \int_a^r f(x)dx\right)$. If the limit **does not exist**, $\int_a^{\infty} f(x)dx$ is divergent. Hence, the area under the curve is undefined.

Similarly,

$$\int_{-\infty}^{b} f(x)dx = \lim_{r\to-\infty} \int_r^b f(x)dx$$

In addition,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

Note: $\int_{-\infty}^{\infty} f(x)dx \neq \lim_{r\to\infty} \int_{-r}^{r} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ is convergent if **BOTH** $\int_{-\infty}^{0} f(x)dx$ and $\int_0^{\infty} f(x)dx$ are convergent.

Ex. 16: Find the area under the curve of function $f(x) = \lambda e^{-\lambda x}$ between $x = [0, \infty)$.

Solution:

$$\text{Area} = \int_0^{\infty} \lambda e^{-\lambda x} dx$$

$$= \lim_{r \to \infty} \int_0^{\infty} \lambda e^{-\lambda x} dx$$

$$= \lim_{r \to \infty} \left[-e^{-\lambda x} \right]_0^r$$

$$= \lim_{r \to \infty} \left[\left(-e^{-\lambda r} \right) - \left(-e^{-\lambda (0)} \right) \right]$$

$$= -0 - (-1)$$

$$= 1$$

Ex. 17: Determine $\int_1^{\infty} \frac{1}{x} dx$ if it converges. Otherwise, show that this definite integral is divergent. [Ans: divergent]

Ex. 18: Determine $\int_1^{\infty} \frac{1}{x^2} dx$ if it converges. Otherwise, show that this improper definite integral is divergent. [Ans: 1]

Note: From Ex. 17 and Ex. 18, the formats are similar but Ex. 54 is divergent while Ex. 55 is convergent. Hence, we cannot just tell from looking at $f(x)$!

Ex. 19: Determine $\int_0^{\infty} xe^{-2x} dx$ if it converges. Otherwise, show that this improper definite integral is divergent. [Hint: Use integration by parts]